Continuous Approximation Model for a Demand Responsive Feeder Service with Meeting Points

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1 Introduction

Operating demand-responsive transport (DRT) requires solving the vehicle routing problem (VRP), a combinatorial problem whose complexity grows quickly with the number of passengers. Solving large instances (e.g., more than 100 passengers) is time-consuming, even using advanced meta-heuristics. This is an obstacle to using mathematical and computational frameworks to optimally design DRT systems. Continuous approximations (CAs) are an attractive alternative for addressing these issues, as they provide analytical functions that can be computed efficiently, enabling fast analysis and optimization. Many mathematical models have been developed to describe the aggregate characteristics and key performance indicators (KPIs) of DRT systems, which can broadly be categorized into two approaches:

Analytical Models from 'First Principles' derive formulae for KPIs using geometric probability and queuing theory, without using data for estimation. Considering random customer locations in a compact operational area, two key works are (i) Beardwood *et al.* (1959) who developed an asymptotic formula for the optimal tour length of the travelling salesman problem (TSP), and (ii) Daganzo (1978) who derived models for DRT average cycle time (closely related to average tour length) based on fleet size, demand rate, and system configuration. More recent works have applied similar principles to increasingly elaborate DRT instances.

Empirical Models Estimated Using Data propose analytic formulae to approximate DRT system outputs as continuous functions of key input variables (e.g., number of passengers, operating area). Model parameters are then estimated using simulated or real-world data. Fu (2003) proposed an analytical model for fleet size, which was calibrated using simulation data. Whereas Quadrifoglio & Li (2009) modelled the optimal cycle time of DRT as a feeder service in a rectangular strip, a model since applied to hybrid fixed/DRT systems in Calabrò *et al.* (2023). Chandra & Quadrifoglio (2013) explore alternative models for optimal cycle time, demonstrating that different models perform better for different operating area geometries based on simulation results. Marković *et al.* (2016) applied generalized linear models (GLM) and support vector regression (SVR) to model fleet size, total vehicle travel time and total vehicle KMs as functions of 11 input variables.

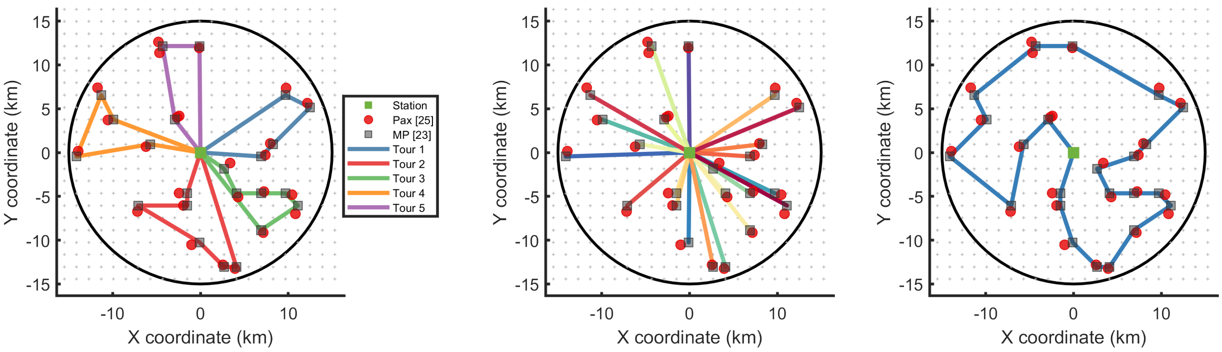
Contribution. This paper adopts the empirical modelling approach to derive a CA model for the total vehicle KMs of a demand responsive feeder service (DRFS) with meeting points (MPs). Meeting-point-based approaches are receiving increasing interest as a means of improving the efficiency of DRT provision. However, MPs introduce additional complexity since the number of pick-up locations is no longer equal to the number of passengers - it depends on the density of passengers and their acceptable walking distance. Distinctively, our model encompasses these critical factors to represent an MP-based service. In addition, whereas most CA models consider only aggregate or averaged quantities, we guarantee level of service at the individual level via a detour factor constraint. We propose a functional form for this CA model and fit it to data from a full DRFS simulation, covering a wide range of scenarios. We show that the resulting CA accurately captures the key characteristics of the system, offering a practical tool for optimizing meeting-point-based DRFS systems.

2 Methodology

We consider a demand-responsive feeder service (DRFS) serving a circular operating area. Passengers (Pax) are picked up from meeting points (MPs) and transported to a transit station at the centre. The operating area, radius R and area $A = \pi R^2$, is covered by a regular grid of potential MPs. Each day N customer requests are generated, uniformly randomly distributed throughout the operating area. The operator assigns each customer to a meeting point (MP) for pick-up, within their maximum walking distance, W. The grid spacing is set as $\sqrt{2}W$ to ensure passengers can reach at least one MP.

Vehicles start from the origin and travel at constant speed along straight lines between visited locations. Vehicles have unlimited range and passenger capacity. Passengers' in-vehicle time is controlled by imposing a constraint on the maximum detour factor, D, experienced by each individual: individual total in-vehicle time can be no more than D times the direct travel time from their pick-up MP to the destination.

A single scenario is defined by $[N, R, W, D]$. Figure 1 shows radius $R = 15km$, within which $N = 25$ passengers (red dots) have been generated at random locations. The central station (green square), regular grid of potential MPs (tiny grey +) and active MPs (23 grey squares) are illustrated. Three solutions are shown, corresponding to different values of D. Passengers submit their ride requests in



(a) $D = 2.5$ *gives 5 tours* (b) $D = 1.0$ *direct taxi* (c) $D = \infty$ *TSP tour*

Figure 1 – *DRFS scenario* $[N, R, W, D] = [25, 15, 1, D]$ *and optimal solutions*

advance and there is no scheduling: all passengers have the same desired arrival time. The operator communicates the MP and pick-up time. No requests are rejected. Passengers are not generated within walking distance of the transit station. Given a set of customer requests, we compute the vehicle tours that minimize the DRFS objective function

$$Z = VTT + PWT \tag{1}$$

the sum of total vehicle travel time (VTT) and passengers' total walking time (PWT). The first term minimises operator costs, the second ensures passengers are assigned to convenient MPs. Details of the problem formulation can be found in Ma *et al.* (2024), along with the metaheuristic algorithm we use to solve this problem. From the algorithm outputs, we focus on total vehicle KMs travelled, V, and number of active MPs, M.

The detour factor constraint forces multiple vehicle tours. For the scenario above the optimal DRFS solution with $D = 2.5$ is shown in Figure (1a): 23 active MPs are covered by 5 vehicle tours. In the extreme case $D = 1$, every passenger must be taken directly from pick-up MP to the destination with no detour. The result is a "direct taxi service" as shown in Figure (1b). Note that multiple passengers may be assigned to the same MP, and hence the number of tours is the number of active MPs, not the number of passengers. The total vehicle KMs in this case will grow linearly with the radius and number of MPs visited $V^{taxi} \simeq RM$. At the other extreme, if the detour constraint is completely relaxed ($D \to \infty$) then any detour will be accepted. The objective function (1) is then minimized by the single tour of the travelling salesman problem (TSP) solution visiting the active MPs; see Figure (1c). When visiting M randomly uniformly distributed locations V^{TSP} grows as $V^{TSP} \simeq RM^{0.5}$.

Model for V: The DRFS solution depends on all scenario inputs $[N, R, W, D]$. However, V depends directly on the number of stops, M, and on R and D. Whereas the walking distance, W, affects the assignment of passengers to MPs, not the subsequent vehicle routing. We therefore seek a nested model $V = V(M, R, D)$ with $M = M(N, R, W)$ since the detour factor will not directly impact on MP assignment. The above limiting cases prompt a model with the following structure:

$$V(M, R, D) = a_1 R M^{0.5} \left(1 + a_2 \frac{M^{a_3}}{D^{a_3}}\right) . \tag{2}$$

Note that in the limit $D \to \infty$ only the leading term remains, matching the TSP formula $V \simeq RM^{0.5}$. In addition, if we set $a_3 = 0.5$ and consider large M with $D = 1$, we get $V \simeq RM$ which would match V^{taxi}. The coefficients will be fitted to data described below.

Model for M: The number of active MPs, M, is an output of the DRFS solution. The operator assigns passengers to MPs up to distance W away. We refer to the walkable area centred on a passenger as a "W-disc". Each W-disc may contain several MPs if $W \gg G$. Again we consider the limiting cases. For small N, passengers are sparse and we expect each passenger will be assigned to their own MP, $M = N$. This can also occur when W-discs are small. For large N, or more precisely, high density, $N/\pi R^2$, most MPs will have passengers assigned. Naively, this could mean we need $M = \pi R^2 / \pi W^2$, the number of W-discs needed to cover the operating area; though since discs tessellate poorly this will be an underestimate. This inspires us to model M with the following structure:

$$M(N, R, W) = \frac{N}{1 + b_1 (\frac{N}{\pi R^2})^{b_2} W^{b_3}} \tag{3}$$

With low density $N/(\pi R^2) \ll 1$, and/or small walking distance, $W \ll 1$, the denominator will collapse to unity and give $M = N$. The terms in the denominator capture the fact that (i) as W increases, passengers walk further and hence can be assigned to fewer MPs, and (ii) as the passenger density increases more passengers will be assigned to the same MPs. For high density, the $N/(\pi R^2)$ term dominates the denominator. With the right coefficients, this model could give the W-disc coverage formula noted above $M = \pi R^2 / \pi W^2$. However, the coefficients will be fitted below.

3 Computational Approach, Results and Discussion

Latin hypercube sampling is used to create a set of 82 scenarios for $[N, R, W, D]$, covering wide ranges for the input variables. See Table 1.

	Passengers (N)	Radius (R)	Walk Max (W)	Detour Factor (D)
Min	20	3	0.7	1.20
Max	500	30	2.00	9.00

Table 1 – *Range of DRFS scenario variables*

Demand Stochasticity: Passenger locations are uniformly randomly generated and hence there can be multiple realizations of demand for one $[N, R, W, D]$ scenario; each demand realization will result in a different DRFS solution, hence different values for M and V. A continuous analytic model for V cannot capture this stochastic variation and will give only a single value for each set of input variables $[N, R, W, D]$ (note the horizontal groups in Figure 2). This *irreducible* variability will be quantified using the mean absolute percentage deviance (MAPD), analogous to the mean absolute percentage error commonly used in modelling. For each of the 82 scenarios we generate 10 random demand instances (days). On each day, N customer locations are generated from the uniform distribution and the DRFS is solved using the meta-heuristic of Ma *et al.* (2024), to provide values for for M and V. Using these data, we first fit the nonlinear model for M from (3), then with this nested inside the model for V we fit (2). Table 2 shows the model fit statistics and coefficient estimates, with MAPD indicating the irreducible variability which accounts for more than half of the mean absolute percentage error (MAPE) in the CA models for M and V.

$$V(M, R, D) = 0.9RM^{0.5} \left(1 + 0.57 \frac{M^{0.48}}{D^{0.64}}\right) \quad \text{with} \quad M(N, R, W) = \frac{N}{1 + 1.99 \left(\frac{N}{\pi R^2}\right)^{0.92} W^{1.78}} \tag{4}$$

The M model for MPs has MAPE = 7.06, with more than half of this amount due to the irreducible variability in MPs (MAPD = 3.79). The nested $V(M)$ model fits well, with MAPE = 8.92; again more

Model	Coeff	Estimate	SE	Statistic	Value
M	b_1	1.99	0.014	MAPD	3.79
	b_2	0.92	0.005	MAPE	7.06
	b_3	1.78	0.01	RMSE	5.14
				R^2	0.996
$V(M)$	a_1	0.90	0.10	MAPD	4.68
	a_2	0.57	0.11	MAPE	8.92
	a_3	0.48	0.02	RMSE	44.0
	a_4	0.64	0.02	R^2	0.997
V_{GLM}	g_0	4.94	0.01	MAPD	4.68
	g_1	0.003	1E-5	MAPE	26.36
	g_2	0.10	0.0002	RMSE	185.92
	g_3	-0.54	0.004	R^2	0.926
	g_4	-0.22	0.002		

Table 2 – *CA models: coefficients, standard errors, and goodness of fit statistics*

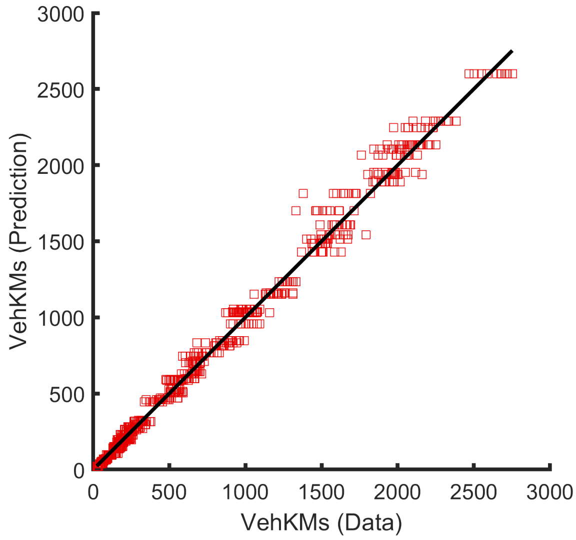


Figure 2 – *Data vs CA Model for VehKMs*

than half of this is accounted for by the MAPD of the data. Recall the idealised limiting behaviours that motivated this model structure. The model for M gives the limit $M = N$ for low passenger density and/or $W \to 0$. With high passenger density we get approximately $M = \pi R^2/2W^{1.78}$. The combined nested model for V in the limit $D \gg 1$ gives the TSP-like model $V = 0.9RM^{0.5}$. For $D = 1$ we have $V = 0.51RM^{0.98}$ which scales nearly linearly with M.

For comparison, we fitted multiple GLM models $V_{GLM} = g_0 + g_1 N + g_2 R + g_3 W + g_4 D$ as used in Marković *et al.* (2016). We considered all common distributions and link functions and all combination of available inputs as linear terms via stepwise regression. The best of these models is reported in Table 2 which achieves only MAPE = 26.36 and RMSE = 185.92.

4 Conclusions

We have proposed an analytical structure that gives a continuous approximation model for the DRFS with meeting points, incorporating an individual passenger-based detour factor constraint to guarantee the level of service. We fit the model based on a set of DRFS scenarios each having 10 random demand realizations. The proposed model outperforms (in terms of errors and applicable range of inputs) other analytical (regression) models proposed in the literature and can readily be used for the design, configuration and decision support of DRFS with meeting points.

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