Continuous-time optimal control for trajectory planning of autonomous vehicles under joint probabilistic constraints.

Ange Valli[0000−0001−9483−6834] and Abdel Lisser[0000−0003−1318−6679]

Laboratoire des Signaux et Systèmes (L2S), Université Paris-Saclay, CNRS, CentraleSupélec, 3 Rue Curie Joliot, 91190, Gif-sur-Yvette, France

{**ange.valli**, **abdel.lisser**}@**l2s.centralesupelec.fr**

Extended abstract submitted for presentation at the 12th Triennial Symposium on Transportation Analysis conference (TRISTAN XII)
June 22-27, 2025, Okinawa, Japan

January 27, 2025

Keywords: Autonomous vehicles, Trajectory planning, Joint probabilistic constraints, Continuous-time optimal control

1 INTRODUCTION

Autonomous vehicle driving should address several challenges related to the safety and comfort of the passenger. Vehicles evolve in uncertain environments, which raises issues in maintaining driving performance. The simulations of autonomous vehicles cannot reproduce the complexity of a real-life environment with all its unexpected events. Our research has been focused on generating reference trajectories for autonomous vehicles. In trajectory planning, a reference trajectory is a predefined prediction of the path of the vehicle computed on a given time horizon. It is used as a benchmark to compare with the simulated path of the vehicle in real-life conditions. During the driving, it compensates for human errors of inattention or judgment by handling constraints which model road conditions and complex situations, such as unexpected behaviour of other vehicles (Dempster *et al.*, 2023).

We developed a joint chance-constrained optimal control model for reference trajectory planning as it presents robustness to stochastic components. The uncertainty due to external factors should be considered in the control. The chance-constrained approach guarantees that a certain level of performance can be expected from a model. We cannot assume to find a feasible solution to the optimal control problem that satisfies all constraints at all time steps because of the stochastic term. Therefore, the chance constraint captures the probability of satisfaction of a constraint and asserts this probability to be higher than a threshold α we determine. Thanks to (Prékopa, 2013) results, we can formulate our chance-constrained optimal control problem as a deterministic equivalent second-order conic-constrained optimal control problem. Our approach does not require relaxation or approximation of the constraints, so we are able to find the exact result rather than bounds for the probability. The robustness of chance constraints is guaranteed by the equivalence between a purely deterministic formulation of the problem and the probabilistic term subject to uncertainty.

2 METHODOLOGY

2.1 Optimal control problem

The optimal control problem in continuous time is described as follows. Let $\mathcal{Z}$ be the feasible set of states, $\mathcal{U}$ be the feasible set of control inputs. Let $z(t) \in \mathcal{Z}$ and $u(t) \in \mathcal{U}$ be the state and

control variables, respectively.
The cost function is an integral between initial and final times t_0 and t_n, with $0 \leq t_0 \leq t_n$, with $\ell : \mathcal{Z} \times \mathcal{U} \longmapsto \mathbb{R}^+$ the integrand. The function designing the system dynamics of the control-state $f : \mathcal{Z} \times \mathcal{U} \longmapsto \mathcal{Z}$ and $c : \mathcal{Z} \times \mathcal{U} \longmapsto \mathbb{R}$ is the inequality constraint function.

$$
\begin{aligned}
\min_{z(\cdot),u(\cdot)} \quad & \int_{t_0}^{t_n} \ell(z(t),\ u(t))dt && (1)\\
\text{s.t.} \quad & \dot{z}(t) = f(z(t), u(t)), && (1a)\\
& c(z(t), u(t)) \leq 0, && (1b)\\
& z\left(t_0\right) = z_{\text{init}}, \quad z\left(t_n\right) = z_{\text{term}}, && (1c)\\
& z(t) \in \mathcal{Z}, \quad u(t) \in \mathcal{U}
\end{aligned}
$$

The constraint (1a) is the control-state equation of the system, (1b) are the constraints on control and state variables and the constraint (1c) defines the initial and terminal states z_{init} and z_{term} of the system.

2.2 Continuous-time chance-constrained reference trajectory generator

The unicycle kinematic model gives the following state of the ego vehicle, the vehicle on which we perform the control, at time t:

$$z_t = [x_t, y_t, \theta_t, v_t]^T \tag{2}$$

where x_t is the longitudinal position, y_t is the lateral position, θ_t is the heading angle of the vehicle and v_t is the linear speed. The control input at time t is given by

$$u_t = [a_t, \omega_t] \tag{3}$$

where a_t is the linear acceleration and ω_t is the angular velocity. The ego vehicle's control-state relationship is given by :

$$\frac{dz_t}{dt} = f(z_t, u_t) \tag{4}$$

where $f(z_t, u_t) = [v_t \cos(\theta_t), v_t \sin(\theta_t), \omega_t, a_t]^T$. The optimal control problem proposed for the reference trajectory generation is the following :

$$
\begin{aligned}
\min_{\mathbf{u},\mathbf{z}} \quad & \int_0^T \mathbf{w}_g * D_t^2(x_t, y_t) + \mathbf{w}_v * (v_r - v_t)^2 + \mathbf{w}_a * a_t^2 \\
& + \mathbf{w}_\omega * \omega_t^2 + \mathbf{w}_j * \left(\frac{da_t}{dt}\right)^2 + \mathbf{w}_h * H(\theta_t)^2 \\
& + \mathbf{w}_p * P(x_t^{tgt}, y_t^{tgt}, x_t, y_t)\ dt && (5)\\
\text{s.t.} \quad & \frac{dz_t}{dt} = f(z_t, u_t), && (5a)\\
& L(x_t, y_t) \quad \leq \quad 0, && (5b)\\
& |v_t| \leq v_{max}, && (5c)\\
& |\omega_t| \leq \omega_{max}, && (5d)\\
& |a_t| \leq a_{max}, && (5e)\\
& \left|\frac{da_t}{dt}\right| \leq j_{max}, && (5f)\\
& K(x_t^{tgt}, y_t^{tgt}, x_t, y_t) \geq d_{min} && (5g)\\
& x_t^{tgt}, y_t^{tgt}, x_t, y_t, v_t \in \mathbb{R}^+ \\
& a_t \in \mathbb{R} \quad \theta_t, \omega_t \in [-\pi, \pi]
\end{aligned}
$$

With $\mathbf{u} = (u(t))_{t\in\mathbb{R}^+}$ and $\mathbf{z} = (z(t))_{t\in\mathbb{R}^+}$ vectors of input control and state, respectively. The quantities x_t^{tgt} and y_t^{tgt} are the target vehicle's longitudinal and lateral positions measured by the ego vehicle. The target vehicle corresponds to a vehicle we do not control, which the ego vehicle must avoid. The distance to the next waypoint at instant t is $D_t^2(x_t, y_t) : \mathbb{R}_+^2 \longmapsto \mathbb{R}^+$. In our study, we consider waypoints on the road's centre lane only. The recommended linear speed value is v_r.
The jerk term is $\left(\frac{da_t}{dt}\right)^2$, which is the linear acceleration rate of change over time. It is responsible for the comfort of the passenger by performing smooth acceleration. A jerk that has a high value can have physiological effects on the human body. We assume the control variable a_t to be differentiable on $[0, T]$.
The distance between the heading angle of the vehicle and the degree of curvature of the centre lane is described by $H(\theta_t) : [-\pi, \pi] \longmapsto \mathbb{R}^+$ and $P(x_t^{tgt}, y_t^{tgt}, x_t, y_t) : \mathbb{R}_+^4 \longmapsto \mathbb{R}^+$ is a potential field function modelling the vehicle's Adaptive Cruise Control (ACC) feature, which adapt the speed of the vehicle when it becomes too close of other vehicles (Jiang *et al.*, 2024).
The function $K(x_t^{tgt}, y_t^{tgt}, x_t, y_t) : \mathbb{R}_+^4 \longmapsto \mathbb{R}^+$ in constraint (5g) represents the distance between ego and target vehicles. The constraint enforces a minimum distance to prevent collisions. The function $L(x_t, y_t) : \mathbb{R}_+^2 \longmapsto \mathbb{R}^+$ and constraint (5b) model the distance between the coordinates of the ego vehicle and the limits of the road.

2.3 Joint probabilistic model

Let's suppose $\forall t \in \mathbb{R}^+ x_t^{tgt} \sim \mathcal{N}(\mu_{x_t}, \sigma_{x_t})$ and $y_t^{tgt} \sim \mathcal{N}(\mu_{y_t}, \sigma_{y_t})$ independent distributions. Constraint (5g) is the stochastic constraint. This assumption models the uncertainty as discussed in Section 1. We formulate the chance constraint such as :

$$\mathbb{P}(|x_t^{tgt} - x_t + y_t^{tgt} - y_t| \geq d_{min}) \geq \alpha \tag{6}$$

Let's consider the new problem with joint probabilistic constraints. Compared to previous research, it considers uncertainty due to both terms $(x_{tgt}^t)_{t\in\mathbb{R}^+}$ and $(y_{tgt}^t)_{t\in\mathbb{R}^+}$ to be covered by the entire probability (Geletu *et al.*, 2013):

$$\mathbb{P}\left(x_t^{tgt} * (-1) \leq -\frac{d_{min}}{\sqrt{2}} - x_t ; y_t^{tgt} * (-1) \leq -\frac{d_{min}}{\sqrt{2}} - y_t\right) \geq \alpha \tag{7}$$

We introduce γ_1, γ_2 artificial variables with $\gamma_1 + \gamma_2 = 1$, $b_t^1 = -\frac{d_{min}}{\sqrt{2}} - x_t$, $b_t^2 = -\frac{d_{min}}{\sqrt{2}} - y_t$ and we obtain the equivalent problem:

$$\mathbb{P}(x_t^{tgt} * (-1) \leq b_t^1)\mathbb{P}(y_t^{tgt} * (-1) \leq b_t^2) \geq \alpha^{\gamma_1+\gamma_2} \tag{8}$$

By studying independently the two constraints

$$\mathbb{P}(x_t^{tgt} * (-1) \leq b_t^1) \geq \alpha^{\gamma_1} \tag{9}$$

$$\mathbb{P}(x_t^{tgt} * (-1) \leq b_t^2) \geq \alpha^{\gamma_2} \tag{10}$$

The deterministic equivalent second-order conic constraints are :

$$F^{-1}(\alpha^{\gamma_1})|\sigma_{x_t}^{\frac{1}{2}}| + \frac{d_{min}}{\sqrt{2}} + x_t \leq \mu_{x_t} \tag{11}$$

$$F^{-1}(\alpha^{\gamma_2})|\sigma_{y_t}^{\frac{1}{2}}| + \frac{d_{min}}{\sqrt{2}} + y_t \leq \mu_{y_t} \tag{12}$$

With the sequential convex approximation algorithm (Scheffe *et al.*, 2022), we can find suitable values for γ_1 and γ_2 to adapt the concentration inequalities to the weight of uncertainty due to stochastic variables $(x_{tgt}^t)_{t\in\mathbb{R}^+}$ and $(y_{tgt}^t)_{t\in\mathbb{R}^+}$.

3 DISCUSSION

In our research, we use chance constraints to obtain a continuous-time approach using dynamic solvers to solve the optimal control problem of trajectory planning for autonomous vehicles. It presents the advantage of a generic model, which means it can be derived for other vehicles by adding constraints for modelling other environments. Those constraints could consider physical parameters of height, depth, wind velocity, or strength of ocean currents to control different types of vehicles such as aircraft, spacecraft, or submarines.

We conducted our study with joint chance constraints to distribute the importance of the constraints with respect to the different stochastic variables (Schmid *et al.*, 2024). Depending on the scenario and the time step, the significance of errors of measurements is due to one source of uncertainty or another. The stochastic model can compensate for one constraint by embedding stochastic components into multiple deterministic equivalent second-order conic constraints to control the distance between vehicles when they get too close. The weights are adapted so the most restrictive component is not violated in the solution of the optimal control problem.

Our model is robust to various types of scenarios. Urban scenarios are handled with good performances, as most classical ones obtain a feasible solution. A solution is considered unfeasible when it is not realistic, which happens when the ego vehicle, the one we control, waits at the beginning of the simulation to have some distance from the target vehicle, the one we avoid. The continuous-time approach is also robust to high-speed highway scenarios, as it represents better real-life conditions.

References

Dempster, Rowan, Al-Sharman, Mohammad, Rayside, Derek, & Melek, William. 2023. Real-time unified trajectory planning and optimal control for urban autonomous driving under static and dynamic obstacle constraints. *Pages 10139–10145 of: 2023 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE.

Geletu, Abebe, Klöppel, Michael, Zhang, Hui, & Li, Pu. 2013. Advances and applications of chance-constrained approaches to systems optimisation under uncertainty. *International Journal of Systems Science*, **44**(7), 1209–1232.

Jiang, Yangsheng, Cong, Hongwei, Chen, Hongyu, Wu, Yunxia, & Yao, Zhihong. 2024. Adaptive cruise control design for collision risk avoidance. *Physica A: Statistical Mechanics and its Applications*, **640**, 129724.

Prékopa, András. 2013. *Stochastic programming*. Vol. 324. Berlin, Germany: Springer Science & Business Media.

Scheffe, Patrick, Henneken, Theodor Mario, Kloock, Maximilian, & Alrifaee, Bassam. 2022. Sequential convex programming methods for real-time optimal trajectory planning in autonomous vehicle racing. *IEEE Transactions on Intelligent Vehicles*, **8**(1), 661–672.

Schmid, Niklas, Fochesato, Marta, Sutter, Tobias, & Lygeros, John. 2024. Joint chance constrained optimal control via linear programming. *IEEE Control Systems Letters*.