Fast heuristic for global optimization of continuous network design problem with stochastic user equilibrium

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Extended abstract submitted for presentation at the 12th Triennial Symposium on Transportation Analysis conference (TRISTAN XII)
June 22-27, 2025, Okinawa, Japan

February 14, 2025

Keywords: continuous network design problem, stochastic user equilibrium, linear programming

1 INTRODUCTION

The continuous network design problem (CNDP) determines the optimal link capacities in a network, considering a traffic assignment principle. CNDP is NP-hard (Gairing, et al., 2017). Hence, we can hardly obtain a globally optimal solution in polynomial time. This has prompted the development of approximation algorithms. Only a few of them, however, focus on the stochastic user equilibrium (SUE) traffic assignment though it is more general than the deterministic user equilibrium (DUE). Firstly, Davis (1994) developed a gradient-based algorithm to obtain a local optimum. Secondly, Liu and Wang (2015) formulated an approximate mixed-integer linear programming (MILP) and solved it exactly. This method obtains a global optimal solution and is often applied to recent studies treating CNDP-like problems, e.g., Zhang et al. (2023). However, the number of integer variables increases when the network is large, or the approximation accuracy becomes high, and the calculation time increases exponentially.

In this paper, we propose a novel algorithm for CNDP with the following advantages: 1) logit-based SUE can be adopted, 2) there is no need to enumerate all paths between an OD pair, 3) no integer variables are included, and 4) the final solution considers travel times without approximation. To achieve these, the algorithm solves a linear programming (LP) formulated by relaxing the original CNDP. After that, it shifts the solution to the LP toward the feasible region of CNDP. Instead of the above advantages, the SUE condition may not hold exactly in the final solution.

The remainder of this paper is organized as follows: Section 2 formulates CNDP with SUE. Section 3 proposes our algorithm illustrating LP to be solved at each iteration. Numerical calculations are conducted to verify the proposed algorithm in Section 4. Section 5 concludes the paper.

2 PROBLEM FORMULATION

Here, we assume a network with node set N and link set A. The sets of origin nodes and destination nodes are denoted as $O(\subset N)$ and $D(\subset N)$, respectively. We aim to minimize the total costs which consists of the total travel cost and the total link enhancement cost. We use the BPR function as the link cost function. We adopt the link-based formulation of SUE proposed in Akamatsu (1996). Therefore, CNDP with SUE, termed [CNDP-SUE], is formulated as follows.

$$\min \sum_{a\in A} (\rho \cdot x_a \cdot t_a(x_a, y_a) + d_a \cdot y_a) \tag{1}$$

with respect to $x_a^r, y_a\ \forall r \in O, \forall a \in A$, subject to

$$t_a(x_a, y_a) = t_a^0 \cdot \left(1 + \alpha_a \cdot \left(\frac{x_a}{C_a + y_a}\right)^4\right) \qquad \forall a \in A \tag{2}$$

$$y_a \geq 0 \qquad \forall a \in A \tag{3}$$

$$x_a = \sum_{r \in O} x_a^r \qquad \forall a \in A(4)$$

$$\sum_{a \in A} x_a^r \cdot \gamma_{a,s} = q_{rs} \qquad \forall r \in O, \forall s \in D(5)$$

$$x_a^r \geq 0, \zeta_a^r \geq 0 \qquad \forall a \in A, \forall r \in O(6)$$

$$x_a^r \cdot \zeta_a^r = \frac{1}{\theta} \cdot x_a^r \cdot l_a^r \qquad \forall a \in A, \forall r \in O(7)$$

where

$$\zeta_a^r = t_a + \sum_{a' \in A} \delta_{a'}^{rs} \cdot t_{a'} - \sum_{a' \in A} \delta_{a'}^{rs'} \cdot t_{a'} \qquad \forall (s,s') = a \in A, \forall r \in O(8)$$

$$l_a^r = h_a^r + \sum_{a' \in A} \delta_{a'}^{rs} \cdot h_{a'}^r - \sum_{a' \in A} \delta_{a'}^{rs'} \cdot h_{a'}^r \qquad \forall (s,s') = a \in A, \forall r \in O(9)$$

$$h_a^r = -\ln \frac{x_a^r}{\sum_{a' \in A} x_{a'}^r \cdot \gamma_{a',s'}^+} \qquad \forall (s,s') = a \in A, \forall r \in O(10)$$

(1) is the objective function, as noted earlier. Its first term represents the total travel cost of link a calculated as the product of its traffic flow x_a, its travel time t_a, and the value of time ρ. The second term represents the enhancement cost of link a, which is calculated as the increase in capacity y_a multiplied by the investment per unit capacity enhancement d_a. A travel time of link a is determined by the BPR function (2), where t_a^0 is the free-flow travel time, C_a is the original capacity, and α_a is a parameter. (4) defines a link flow x_a. x_a^r denotes the flow on link a departed from node r. Similarly, (5) defines an OD demand q_{rs}. The constant $\gamma_{a,s}$ is defined as $\gamma_{a,s}^+ - \gamma_{a,s}^-$ for $\forall (i,j) = a \in A, \forall s \in N$, where $\gamma_{a,s}^+$ equals 1 if $s = j$, and 0 otherwise, and $\gamma_{a,s}^-$ equals 1 if $s = i$, and 0 otherwise. (6) and (7) represent the complementarity condition of SUE. θ is a parameter, and ζ_a^r and l_a^r are variables defined by (8)-(10). Here, δ_a^{rs} equals 1 if link a is included in the shortest path between the OD pair r, s after the traffic assignment, and 0 otherwise. When the complementarity condition holds, all path costs between an OD pair r, s are equal, where the path cost is calculated by assuming the cost of link a is $t_a - h_a^r/\theta$.

3 SOLUTION ALGORITHM

[CNDP-SUE] is nonlinear and nonconvex due to the link cost function (2) and the complementarity condition (7). To make it solvable, we formulate the following linearly relaxed CNDP, termed [LR-CNDP]$_0$.

[LR-CNDP]$_0$

$$\min \sum_{a \in A} (\rho \cdot (\mu_a + \varepsilon_z \cdot \nu_a) + d_a \cdot y_a) \qquad (11)$$

with respect to x_a^r, y_a, μ_a, ν_a $\forall r \in O, \forall a \in A$, subject to (3) – (6),

$$(x_a, y_a, \mu_a) \in \mathrm{PL}(\mathrm{Conv}(\mathrm{M}_a)) \qquad \forall a \in A(12)$$

$$(x_a, y_a, \nu_a) \in \mathrm{PL}(\mathrm{Conv}(\mathrm{N}_a)) \qquad \forall a \in A(13)$$

where

$$\mathrm{M}_a = \{(x_a, y_a, z) | z \geq (x_a - \varepsilon_z) \cdot t_a(x_a, y_a)\}, \mathrm{N}_a = \{(x_a, y_a, z) | z \geq t_a(x_a, y_a)\} \qquad \forall a \in A(14)$$

$$\zeta_a^r = \nu_a + \sum_{a' \in A} \delta_{a'}^{rs} \cdot \nu_{a'} - \sum_{a' \in A} \delta_{a'}^{rs'} \cdot \nu_{a'} \qquad \forall (s,s') = a \in A, \forall r \in O(15)$$

Here, δ_a^{rs} is calculated based on the shortest path when $t_a = t_a^0$ for all $a \in A$. Conv$(\cdot)$ is an operator which forms the convex hull of a given set, and PL$(\cdot)$ is an operator which forms the convex polytope obtained by piecewise linearizing a given convex set from the outside. The feasible region of μ_a is enclosed by tangent planes of $z = (x_a - \varepsilon_z) \cdot t_a(x_a, y_a)$ calculated at different points of (x_a, y_a). In a similar way, the feasible region of ν_a is enclosed by tangent planes of $t_a(x_a, y_a)$ calculated at different points of (x_a, y_a). There two regions include nonconvex sets M_a and N_a, respectively. By minimizing the objective function (11), it is expected that the following two relationship, i.e., $\mu_a \approx (x_a - \varepsilon_z) \cdot t_a(x_a, y_a)$ and $\nu_a \approx t_a(x_a, y_a)$, hold. When a small value $\varepsilon_z > 0$ approaches zero, $(x_a - \varepsilon_z) \cdot$

$t_a(x_a, y_a) = x_a \cdot t_a(x_a, y_a)$ is obtained. Therefore, $\mathrm{Conv}(\mathrm{M}_a) \approx \mathrm{M}_a$. Moreover, μ_a can be minimized without being affected by any other constraints than (12). Hence, $\mu_a \approx (x_a - \varepsilon_z) \cdot t_a(x_a, y_a)$ is always true by obtaining many tangent planes of $(x_a - \varepsilon_z) \cdot t_a(x_a, y_a)$ and by defining the convex polytope consisting of them as the feasible region of μ_a. In contrast, ν_a cannot be approximately calculated like μ_a because the link cost function (2) is nonconvex. Moreover, we do not consider the complementarity condition (7) here. Therefore, there may exist some $a \in A$ and $r \in O$ that do not satisfy (7).

To shift the infeasible solution of the original CNDP obtained above gradually to the feasible region, we will iteratively solve the following LP, termed [LR-CNDP] $_{i+1}$, which is the problem solved at the $(i+1)$th iteration. [LR-CNDP] $_{i+1}$ is formulated using the solution to [LR-CNDP] $_i$. The solution to [LR-CNDP] $_i$ is denoted by the variable, that is addressed in the original problem and in (4), (8) and (9), to which the subscript of i is added, i.e., $x^r_{a,i}, y_{a,i}, x_{a,i}, \zeta^r_{a,i}, l^r_{a,i}$.

[LR-CNDP] $_{i+1}$

$$\min \sum_{a \in A} (\rho \cdot (\mu_a + \varepsilon_z \cdot \nu_a) + d_a \cdot y_a) + E \cdot \sum_{a \in A} \left(\lambda^+_a + \lambda^-_a + \sum_{r \in O} \lambda^r_a \right) \quad (16)$$

with respect to $x^r_a, y_a, \mu_a, \nu_a, \lambda^+_a, \lambda^-_a, \lambda^r_a \; \forall r \in O, \forall a \in A$, subject to (3) – (6), (12), (13),

$$p_{a,i} - (1 - \varepsilon_p)^{i+1} \cdot L^-_a - \lambda^-_a \le \nu_a \le p_{a,i} + (1 - \varepsilon_p)^{i+1} \cdot L^+_a + \lambda^+_a \qquad \forall a \in A (17)$$

$$\lambda^+_a \ge 0, \lambda^-_a \ge 0, \lambda^r_a \ge 0 \qquad \forall a \in A, \forall r \in O (18)$$

if $t_{rs,i} < t_{rs',i}$

$$\zeta^r_a = \begin{cases} -\dfrac{\zeta^r_{a,i}}{x^r_{a,i}} \cdot x^r_a + 2 \cdot \sqrt{\dfrac{\zeta^r_{a,i}}{x^r_{a,i}} \cdot \Phi^r_{a,i}} + \lambda^r_a & \text{if } \zeta^r_{a,i} > \dfrac{l^r_{a,i}}{\theta} \\ -\dfrac{\zeta^r_{a,i}}{x^r_{a,i}} \cdot x^r_a + 2 \cdot \sqrt{\dfrac{\zeta^r_{a,i}}{x^r_{a,i}} \cdot \Phi^r_{a,i}} - \lambda^r_a & \text{if } \zeta^r_{a,i} < \dfrac{l^r_{a,i}}{\theta} \end{cases} \qquad \forall (s, s') = a \in A, \forall r \in O (19)$$

$$x^r_a \ge \varepsilon_x \qquad \forall (s, s') = a \in A, \forall r \in O (20)$$

otherwise

$$x^r_a = 0 \qquad \forall (s, s') = a \in A, \forall r \in O (21)$$

where

$$p_{a,i} = \left.\frac{\partial t_a}{\partial x_a}\right|_{x_{a,i}, y_{a,i}} \cdot (x_a - x_{a,i}) + \left.\frac{\partial t_a}{\partial y_a}\right|_{x_{a,i}, y_{a,i}} \cdot (y_a - y_{a,i}) + t_a(x_{a,i}, y_{a,i}) \qquad \forall a \in A (22)$$

$$\Phi^r_{a,i} = (1 - \varepsilon_c) \cdot x^r_{a,i} \cdot \zeta^r_{a,i} + \varepsilon_c \cdot x^r_{a,i} \cdot \frac{l^r_{a,i}}{\theta} \qquad \forall a \in A, \forall r \in O (23)$$

ν_a in (17) approximately represents the link cost function (2). (22) represents the tangent plane of the link cost function at $(x_a, y_a) = (x_{a,i}, y_{a,i})$. Positive constants L^+_a and L^-_a are introduced to (17) to ensure that the feasible region of ν_a narrows down to (22) as i increases, where $\varepsilon_p > 0$ is a small positive constant. λ^+_a, λ^-_a and λ^r_a are introduced to respective constraints to make the solution feasible, and the sum of them is minimized by (16) accompanied by a positive large coefficient E. By (19), a solution that approximately satisfies the complementarity condition (7) can be obtained. (23) denotes the internally dividing point of $x^r_{a,i} \cdot \zeta^r_{a,i}$ and $x^r_{a,i} \cdot l^r_{a,i}/\theta$ in the ratio of ε_c to $1 - \varepsilon_c$. When ε_c in (23) equals zero, the first and second terms on the right-hand side of (19) represent the tangent plane of $x^r_a \cdot \zeta^r_a = x^r_{a,i} \cdot \zeta^r_{a,i}$ calculated at $(x^r_a, \zeta^r_a) = (x^r_{a,i}, \zeta^r_{a,i})$. By introducing a small positive value $\varepsilon_c > 0$, $(x^r_{a,i+1}, \zeta^r_{a,i+1})$ shifts to $x^r_a \cdot \zeta^r_a = x^r_{a,i} \cdot l^r_{a,i}/\theta$. When $x^r_{a,i}$ in (10) equals zero, however, ln 0 diverges and therefore $l^r_{a,i}$ in (9) cannot be defined. Due to this, we need to address only the paths that carry flows. For this, we applied the criteria employed in Dial's algorithm (Dial, 1971). Regarding link $a = (s, s')$ and node r, when the shortest travel time between the OD pair r, s obtained at the ith iteration, denoted by $t_{rs,i}$, is shorter than $t_{rs',i}$, both (19) and (20) hold where $\varepsilon_x > 0$ is a small positive constant, otherwise (21) holds.

Note that δ^{rs}_a must be recalculated if necessary. To determine whether to recalculate or not, we use the criterion $\Phi^r_{a,i} < 0$. This holds when $l^r_{a,i}$ is a negative number with a large absolute value and $\zeta^r_{a,i}$ takes a value which is close to zero. In such a case, the path that contains link a has almost the same travel time as the original shortest path and carries much flow. In the case of SUE assignment, the path with the largest flow can be the shortest path. Thus, this criterion makes sense.

4 NUMERICAL CALCULATION

Firstly, we solved CNDP with UE by setting $\theta \rightarrow \infty$ and verified our algorithm by comparing it with the results shown in Wang and Lo (2010). They demonstrated that an exact solution to the CNDP can be obtained by solving the MILP, which is an approximate problem of the CNDP. We used totally the same instant as theirs (16-arc network, two scenarios of OD demand). Traffic flows satisfying the complementarity condition (7) could finally be obtained with errors below 10^{-4} in 4.55 seconds for Scenario I and in 3.22 seconds for Scenario II. Table 1 presents the obtained objective values. Our algorithm provided almost the same solution in a short time as the exact one.

Secondly, we solved CNDP with SUE by changing the value of θ for Scenario II of the above problem. It took about 2 minutes to solve each problem. Figure 1 shows the relationship between the value of θ and the value of the objective function (16). The blue dot in the figure shows that the solution that holds the complementary condition (7) was obtained. The orange dots show that such solutions were not obtained. The complementary condition does not hold in some cases where the value of θ approaches zero. This is because the target of convergence $x_{a,i}^r \cdot l_{a,i}^r/\theta$ fluctuates significantly between iterations even when solution x_a^r, ζ_a^r changes little.

Table 1 – *Comparison of our algorithm and the previous one*

Scenario	Our algorithm	Wang and Lo (2010)
I	199.64	199.63
II	522.62	522.59

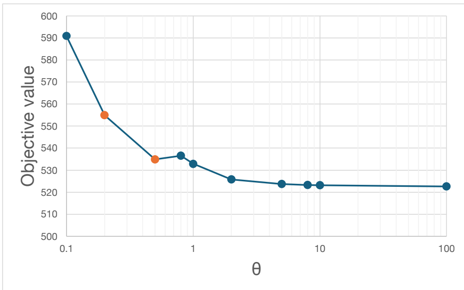

Figure 1 – *Result of [CNDP-SUE]*

5 CONCLUSION

We have proposed a fast solution algorithm for CNDP with SUE. This algorithm is not exact, and the SUE condition may not exactly hold in the final solution. However, numerical calculations indicated that an accurate assignment was highly likely to succeed when θ was large, allowing us to obtain a solution very close to the optimal one in such a case. Developing an algorithm that exactly satisfies the SUE condition even if θ is small reminds a future challenge.

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