# Nonlinear and Ready-to-depart Based Bus Holding Control

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# 1 INTRODUCTION

As urbanization escalates, efficient public transportation is crucial to reduce congestion and environmental impacts. Buses are vital but often suffer from poor reliability due to a vicious cycle: delays cause longer dwell times at stops, leading to further delays and bus bunching (Newell and Potts, 1964). This results in unpredictable services, extended passenger wait times, and overcrowding. Bus holding strategies aim to break this cycle. *Traditional schedule-based holding* uses preset slacks at control points, holding early buses until their scheduled departure times. However, it was commonly believed to require excessive slack time and struggles with significant disruptions (e.g., Daganzo, 2009).

Daganzo (2009) introduced a forward-headway control (FHC) using forward arrival headways, improving reliability over traditional methods. Xuan et al. (2011) generalized it and proposed a near-optimal "Simple Control (SC)" strategy that minimizes holds within schedule deviation thresholds. However, these linear holds still necessitate considerable slacks. A simple adjustment to these linear holds, setting negative holds to zero, introduces nonlinearity but potentially reduces holding times. The effectiveness of such nonlinear controls remains underexplored.

Moreover, previous studies often neglect real-world complexities like random passenger arrivals and variable boarding times. They typically overlook passengers arriving during dwell times and rely on outdated arrival data rather than "ready-to-depart" information.

This paper addresses these challenges through simulation by evaluating nonlinear adaptations of established linear controls, assessing inaccuracies from misrepresented boarding dynamics, and developing a control law informed by "ready-to-depart" data.

# 2 NONLINEAR HOLDING CONTROL

## 2.1 Linear holding controls and their nonlinear adaptations

We analyze a linear bus route assuming: (i) buses depart the terminal with uniform headway $H$; (ii) buses have unlimited capacity; and (iii) buses can be held at any stop following passenger boarding.

The scheduled bus motion model is as follows:

$$t_{n,s+1}^{a} = t_{n,s}^{a} + \lambda_s \tau H + E_s(l_{n,s}) + c_s \quad (1)$$

$$t_{n+1,0}^{a} = t_{n,0}^{a} + H,\ t_{0,0} = 0 \quad (2)$$

where $t_{n,s}^{a}$ denotes the scheduled arrival time of bus $n = 0,1,\ldots$ at stop $s = 0,1,\ldots$; $c_s$ the scheduled travel time between stops $s$ and $s+1$; $\lambda_s$ the passenger arrival rate at stop $s$; $\tau$ the mean boarding time per passenger; and $E_s(l_{n,s})$ the slack (expected holding time per bus) at stop $s$. Here, $n = 0$ corresponds to a virtual bus that adheres to the schedule; and $s = 0$ indicates the terminal.

The actual bus motions are modeled as:

$$a_{n,s+1}^{a} = a_{n,s}^{a} + b_{n,s} + l_{n,s} + c_s + v_{n,s+1} \quad (3)$$

$$a_{n,0}^{a} = t_{n,0}^{a}, \text{ and for the virtual bus, } a_{0,s}^{a} = t_{0,s}^{a} \quad (4)$$

where $a_{n,s}^{a}$ denotes the actual arrival time of bus $n$ at stop $s$; $b_{n,s}$ the actual dwell time; and $v_{n,s+1}$ a normal random noise with mean 0 and standard deviation (SD) $\sigma_s$.

Subtracting (1) from (3), we have the arrival schedule deviation $\varepsilon_{n,s+1}^{a} = a_{n,s+1}^{a} - t_{n,s+1}^{a}$ as:

$$\varepsilon_{n,s+1}^{a} = \varepsilon_{n,s}^{a} + (b_{n,s} - \lambda_s \tau H) + \left(l_{n,s} - E_s(l_{n,s})\right) + v_{n,s+1} \quad (5)$$

Prior research (Daganzo, 2009; Daganzo and Pilachowski, 2011; Xuan et al., 2011) posits that $b_{n,s} = \lambda_s \tau h_{n,s}^{a}$ where $h_{n,s}^{a} = a_{n,s}^{a} - a_{n-1,s}^{a} = \varepsilon_{n,s}^{a} - \varepsilon_{n-1,s}^{a} + H$ is the forward arrival headway.

This implies: (i) passengers arriving during dwell time cannot board the present bus; (ii) passenger arrivals are deterministic; and (iii) a constant boarding time $\tau$ is assumed. We adhere to these simplifying assumptions in this section, leading to the reformulation of Eq. (5):

$$\varepsilon_{n,s+1}^{a} = \varepsilon_{n,s}^{a} + \lambda_s \tau\left(h_{n,s}^{a} - H\right) + \left(l_{n,s} - E_s\left(l_{n,s}\right)\right) + v_{n,s+1} \tag{6}$$

Holding time in Xuan et al. (2011) is determined by:

$$l_{n,s} = E_s\left(l_{n,s}\right) - \varepsilon_{n,s}^{a} + \lambda_s \tau\left(H - h_{n,s}^{a}\right) + \sum_i f_i \cdot \varepsilon_{n-i,s}^{a} \tag{7}$$

where $f_i$'s are control coefficients. To ensure a nonnegative $l_{n,s}$, slack $E_s\left(l_{n,s}\right)$ is set to be sufficiently large, e.g., three SDs of term $\left(\varepsilon_{n,s}^{a} - \lambda_s \tau\left(H - h_{n,s}\right) - \sum_i f_i \cdot \varepsilon_{n-i,s}^{a}\right)$. Note that (7) utilizes only the bus arrival-time data, therefore termed an "arrival-based control." Also note that various existing controls are special cases of (7): *FHC* sets $f_0 = 1 - \alpha$, $f_1 = \alpha$; the backward-headway control (*BHC*) in Bartholdi and Eisenstein (2021) entails $f_{-1} = \alpha$, $f_0 = 1 + \lambda_s \tau - \alpha$, $f_1 = -\lambda_s \tau$; the Eulerian two-way looking headway control (*TWHC*) in Daganzo and Pilachowski (2011) uses $f_{-1} = f_1 = \alpha$, $f_0 = 1 - 2\alpha$; and *SC* uses $f_0 = \alpha \in [0,1)$. The rest $f_i = 0$ in the above special cases. Note the *traditional schedule-based holding* is a special SC with $\alpha = 0$, setting all $f_i = 0$.

A nature treatment to reduce slacks is to incorporate a nonlinear $[x]^{+} = \max\{0, x\}$ into (7):

$$l_{n,s} = \left[D_s - \varepsilon_{n,s}^{a} + \lambda_s \tau\left(H - h_{n,s}^{a}\right) + \sum_i f_i \cdot \varepsilon_{n-i,s}^{a}\right]^{+} \tag{8}$$

(8) is guaranteed to be nonnegative and $D_s$ emerges as an additional control parameter.

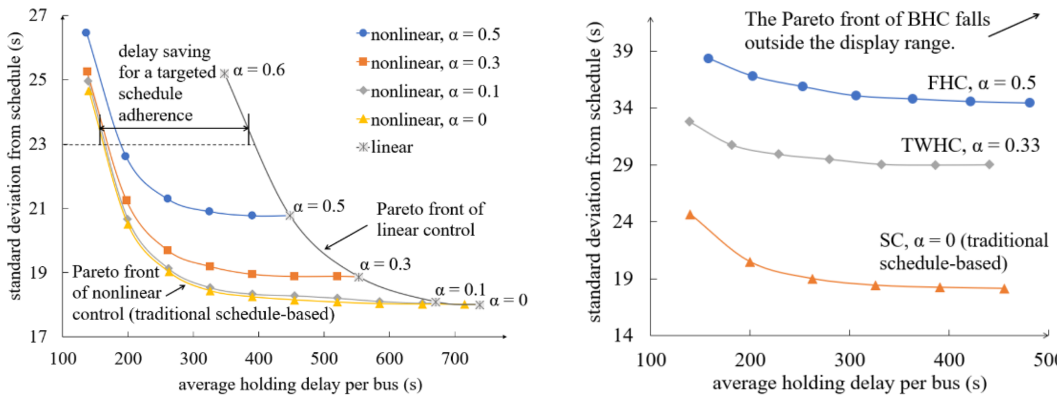
## 2.2 Pareto-optimal holds for schedule deviation and holding delay

We investigate the Pareto efficiency of nonlinear holds aimed at improving schedule adherence while reducing holding delays. Schedule adherence enhances bus fleet management and can lower operational costs. We simulate homogeneous bus systems under various controls given $H = 10$ min, $\tau = 2$ s, $\lambda_s = 1.5$ passengers/min, and $\sigma_s = 18$ s, $\forall s$. Each run involves 5000 buses serving a route with 12 stops. A minimum of 10 runs were executed to guarantee result convergence.

Figure 1a plots the standard deviation from schedule ($\sqrt{var(\varepsilon_{n,s})}$) at the last stop against average holding delay per bus under nonlinear SC with varying $D_s \equiv D$ and $\alpha$ values. The Pareto front (PF) of the linear SC is provided for comparison. Major findings include:

1. Schedule deviation decreases toward the value of linear SC as holding delay increases.
2. Nonlinear SC significantly reduces delays without sacrificing schedule adherence.
3. An optimal $\alpha^* \approx 0$ is observed for nonlinear SC, implying that *traditional schedule-based holding tends to be Pareto efficient*, challenging the prevalent doubts regarding its effectiveness.

Figure 1b shows PFs for nonlinear variants of FHC, BHC, TWHC, and SC—each optimized at its own $\alpha^*$. BHC performs poorly even with perfect backward headway prediction. Results confirm that nonlinear SC outperforms other controls, consistent with observations for linear controls. Thus, *traditional schedule-based holding emerges as the most effective strategy* among them.




(a) Nonlinear versus linear Simple Control (b) PFs of various nonlinear controls

Figure 1 – Pareto-optimal control minimizing schedule deviation and holding delay

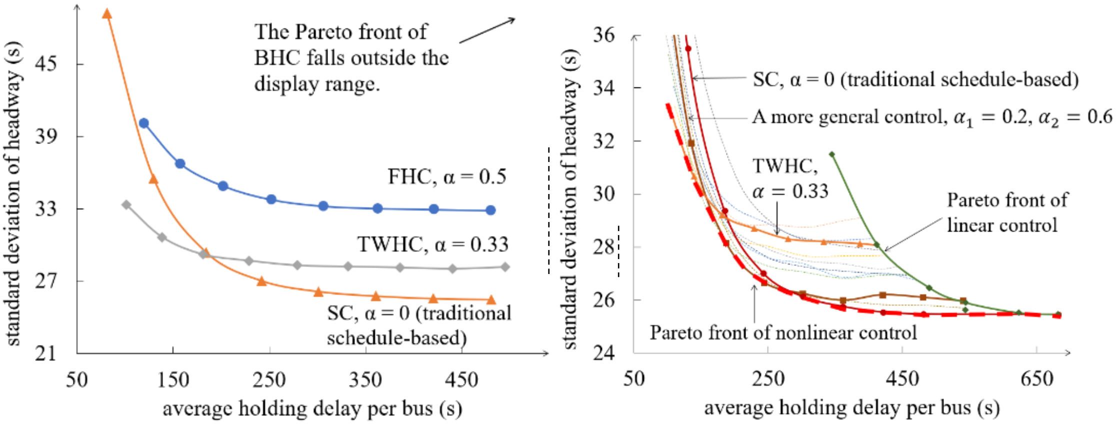
## 2.3 Pareto-optimal holds for headway variation and holding delay

Passengers may prefer consistent headways over strict schedule adherence, as it directly affects waiting times. Figure 2a shows that when measuring headway variation against holding delay, the performance curves for TWHC (assuming accurate backward headway prediction) and SC intersect, meaning that the Pareto-efficient control alternates between TWHC and SC, unlike in Figure 1b.

The non-smooth lower envelope in Figure 2a suggests there may exist a control better than TWHC and SC. We propose a general two-way-looking control defined as $f_{-1} = f_1 = \alpha_1$, $f_0 = \alpha_2 - 2\alpha_1$, where $0 \le \alpha_1 \le \alpha_2 < 1$. It reduces to TWHC for $\alpha_2 = 1$ and to SC for $\alpha_1 = 0$.

Figure 2b displays headway variation versus holding delay for various $(\alpha_1, \alpha_2)$ pairs. The PF, shown as the bold dashed curve—the lower envelope of these plots—indicates TWHC with $\alpha = 0.33$ excels for delays under 14 seconds per stop, SC with $\alpha = 0$ prevails over 23 seconds, and our new control with $\alpha_1 = 0.2, \alpha_2 = 0.6$ is optimal in between.

For comparison, the PF for linear controls with only $f_{-1}, f_0, f_1$ being nonzero, is also included.




(a) PFs of various controls (b) PFs of nonlinear and linear controls

Figure 2 – Pareto-optimal control minimizing headway variation and holding delay

# 3 READY-TO-DEPART BASED HOLDING CONTROL

We propose a ready-to-depart (RTD) based hold that uses *information known at the RTD time (door-closing time)* right before hold is executed. Contrary to arrival-based holds, RTD-based hold knows the exact dwell time, therefore eliminating the impacts of overlooking random passenger arrivals, variable boarding times, and the passengers arriving during the dwell time who should be admitted onboard. Furthermore, the RTD hold is notably simpler.

Let $a^d_{n,s}$ and $t^d_{n,s}$ represent the actual and scheduled departure times post-holding, and $a^r_{n,s}$ and $t^r_{n,s}$ the actual and scheduled RTD times. The deviations from the schedule at departure and RTD are $\varepsilon^d_{n,s} = a^d_{n,s} - t^d_{n,s}$ and $\varepsilon^r_{n,s} = a^r_{n,s} - t^r_{n,s}$, respectively. The model is presented as follows:

$$a^d_{n,s} = a^r_{n,s} + l_{n,s} \tag{9}$$

$$t^d_{n,s} = t^r_{n,s} + E_s(l_{n,s}) \tag{10}$$

$$\varepsilon^d_{n,s} = \varepsilon^r_{n,s} + l_{n,s} - E_s(l_{n,s}) \tag{11}$$

We define the nonlinear holding time $l_{n,s}$ as follows:

$$l_{n,s} = \left[D_s - \varepsilon^r_{n,s} + F_{n,s}\right]^+ \tag{12}$$

where $F_{n,s} = \sum_i \left(f^a_i \cdot \varepsilon^a_{n-i,s} + f^r_i \cdot \varepsilon^r_{n-i,s} + f^d_i \cdot \varepsilon^d_{n-i,s}\right)$, and $f^a_i$, $f^r_i$ ($i$ takes any integer value such that $n - i$ is a valid bus index), and $f^d_i$ ($i \ge 1$) are control coefficients. In turn, we have:

$$\varepsilon^d_{n,s} = \max\{D_s + F_{n,s}, \varepsilon^r_{n,s}\} - E_s(l_{n,s}) \tag{13}$$

The linear form of Eq. (13) is succinctly expressed as $\varepsilon^d_{n,s} = F_{n,s}$. Notably, the control law in Eq. (12) omits dwell time $b_{n,s}$, as well as information regarding demand rate $\lambda_s$, passenger arrival patterns, and boarding durations, which simplifies the data requirements for control implementation.

In the case where $f_i^r = f_i^d = f_i^a = 0, \forall i$, (13) simplifies to $\varepsilon_{n,s}^d = \max\{D_s, \varepsilon_{n,s}^r\} - E_s(l_{n,s})$, and its linear form is reduced to $\varepsilon_{n,s}^d = 0$. *This RTD-based version of traditional schedule-based control is indeed the one used in practice, which holds buses until a predetermined departure time*.

Figure 3 demonstrates the efficacy of RTD-based control with simulation parameters set to $H = 10$ min, $\lambda_s = 3$ passenger/min, $\sigma_s = 12$ s, $\tau = 2$ s, and $\sigma_\tau = 0.5$ s. The figure clearly indicates that RTD-based control substantially decreases the variability in both the arrival and departure headways of buses.

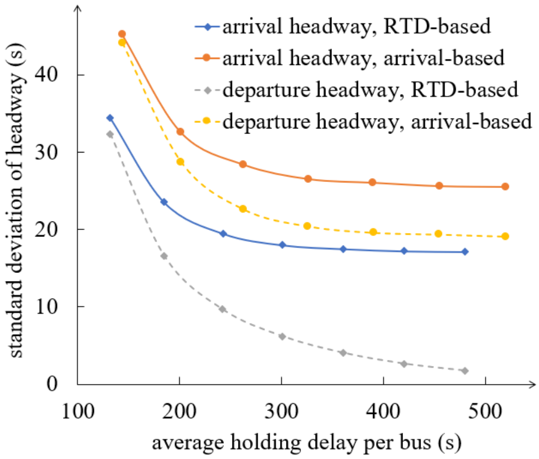



Figure 3 – Benefits of RTD-based control

# 4 CONCLUSIONS

We investigated two simple ideas: nonlinear holding control using a ramp function, and RTD-based control leveraging the latest information when a bus is ready to be held. Despite their simplicity, these methods yield results that diverge from existing literature, offering valuable insights.

Our analysis reveals that the nonlinear variant of *traditional schedule-based holding surprisingly achieves the highest Pareto efficiency*, minimizing schedule deviations for any threshold of holding time. Even when prioritizing headway stabilization, traditional holding proves superior with sufficient allowable delay, contradicting common assertions. We further introduce a *novel control extending SC and TWHC that surpasses both for headway stabilization*. To address real-world complexities, we propose *an RTD-based control using the most recent departure information*, eliminating the need to estimate dwell time and simplifying implementation without demand estimates. Nonlinear RTD-based control reduces holding delays without sacrificing stability, outperforming linear and arrival-based controls.

Our findings benefit transit operators, as the *nonlinear RTD-based traditional holding* addresses critiques of prolonged slacks and delays and is straightforward to implement.

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