Complex dynamics in transportation networks in the context of assignment

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1 INTRODUCTION

The focus of the paper is on the medium term dynamics of transportattion networks in the context of DTA (Dynamic Traffic Assignment), in particular the formation of a network equilibrium.

The context of this question is evolving due to technological progress. Travellers benefit from new means of information (crowd sourcing, internet operators, connectivity of vehicles), which can inform them on the current network state nearly instantaneously. As a result, travellers interact over the whole range of the network indirectly, through the agency of the information systems. Do the route choices carried out in these conditions lead to network equilibria? Instabilities must be expected, if only because traffic conditions change while travellers are moving and they will not meet the conditions upon which they based their decisions. In the future, the emergence of automated vehicles, expected to be controlled in order to optimize their passengers' trips, will add another level of complexity in the transportation system. The day-to-day process of departure time and route choice process should be impacted, and it is not obvious whether networks will function at equilibrium over medium range periods of weeks or months.

The properties of network equilibria in the context of traffic assignment have been studied for a long time. Unicity of equilibria in the static case and simple behavioral asumptions is well-known. Under the same asumptions traffic on multi-modal networks usually admit several equilibria, some stable and some unstable. Early works pointed out the possibility of more complex, even chaotic dynamics (Guo & Huang, 2009), (Han *et al.* , 2011). More recently (Cantarella & Fiori, 2022) carried out more complete analyses of the possibility of such complex behavior in the static case. Less is known in the dynamic case, because of the inherent difficulty of DTA. Let us mention Iryo (2019), as well as (Liu *et al.* , 2018) who considered the possibility of chaotic behavior in a quasi-static context (two departure times).

This paper shows that the results obtained in the static and quasi-static context carry over to the dynamic case and the paper proposes a framework for addressing such problems.

2 METHODOLOGY

The main tool for this study is the GSOM model Lebacque & Khoshyaran (2018), the basic features of which are the following: it is macroscopic, with a conservation law for the vehicular flow completed by a behavioral law (fundamental diagram), vehicular and passenger attributes are advected, and the model admits extensions (multimodal systems, bidimensional representation of networks).The model admits analytical representations and allows for fast calculation of network flows.

The main idea for addressing DTA problems is to consider that i) route choice is carried out by travellers on the basis of their instantaneous travel costs ITC (travel costs include travel times, financial costs and penalty for early/late arrival time), ii) travellers carry out departure time choices based on their predictive travel costs PTC. Thus the time scales involved differ: the time scale of route choice is nearly instantaneous, resulting from the current state of the network. The departure time choice is carried out on a day to day basis based on experience and learning. The total OD demands, $D_w^{t_a}$, given for all OD (origin-destination couples) $w \in \mathcal{W}$ and all desired arrival times $t_a \in \mathcal{T}_a$, constitute the main data of the DTA problem. The route choice proportions $\varpi_p^{t_a}(t)$, for all relevant OD paths $p \in \mathcal{P}_w$, all ODs $w \in \mathcal{W}$ and all times $t \in \mathcal{T}_d$ within the day, and the distribution of departure times, $\varphi_p^{t_a}(t)$, for all ODs $w \in \mathcal{W}$ and all times $t \in \mathcal{T}_d$, constitute the unknowns of the problem.

The within the day dynamics of the system can be summarized by $\dot{X}(t) = \mathcal{F}(\varphi, \varpi, X(t))$ with $X(t)$ the state of the network at time t. The GSOM model provides this expression $\mathcal{F}$. The route choice ϖ results from the current state $X(t)$ via the instantaneous travel times, from which the instantaneous travel costs are deduced, thus $ITC(t) = \mathcal{I}(X(t))$ and $\varpi(t) = \Omega(ITC(t))$ (for instance by a Logit model), i.e. $\varpi(t) = \Omega(\mathcal{I}(X(t)))$. We can summarize this whole process as $X = \mathcal{X}(\varphi, \varpi, X_0)$. From these dynamics the experience travel times are observed, from which the predictive travel times and travel costs are deduced: $PTC = \Pi(X)$. Finally the predictive travel times yield the departure time distribution: $\varphi = \Phi(PTC)$. The network equilibrium is defined by

$$\left| \begin{array}{l} X = \mathcal{X}(\varphi, \varpi, X_0) \\ \varpi = \Omega(\mathcal{I}(X)) \\ PTC = \Pi(X) \\ \varphi = \Phi(PTC) \end{array} \right. \tag{1}$$

Since all functions in (1) are continuous and the unknowns φ, ϖ are defined on closed bounded convex sets (they are probability distributions), it follows by Brouwer or Kakutani type arguments that equilibria solutions of (1) exist.

In a day-to-day iterative process travellers carry out their choice based on experience and learning. Let τ denote the day/iteration index. The day-to-day iterations yield successive estimates $\varphi^\tau, \varpi^\tau$ and corresponding network states X^τ and predictive travel costs PTC^τ. The most basic conceivable iteration would be $PTC^{\tau+1} = \Pi(X^\tau)$, $\varphi^{\tau+1} = \Phi(PTC^{\tau+1})$, i.e. with no learning. This scheme does not converge, nor does it represent traveller behavior. In numerical applications we use schemes of the general form

$$\begin{array}{l} PTC^{\tau+1} = p\left(\Pi(X^\tau), PTC^\tau\right) \\ \varphi^{\tau+1} = f\left(\Phi(PTC^{\tau+1}), \varphi^\tau\right) \end{array} \tag{2}$$

with p and f linear smoothing schemes. Many other learning schemes can be defined, for instance over several iterations. Also learning behavior is liable to concern ϖ.

3 RESULTS

Most studies reported here were made on a large network (15 × 25 km) west of the Ile-de-France region in which we consider 4 main OD couples and the morning peak (7.30-11 am). The traffic is modelled by GSOM. In the cases reported here we used Logit assignment and simple learning strategies (linear smoothing) as in (Liu *et al.* , 2018) and (Cantarella & Fiori, 2022).

Multiple equilibria can occur. Figure 1 illustrates such an occurrence. The figure depicts the dynamics of the systems under two different learning strategies: i) with φ and PTC learning

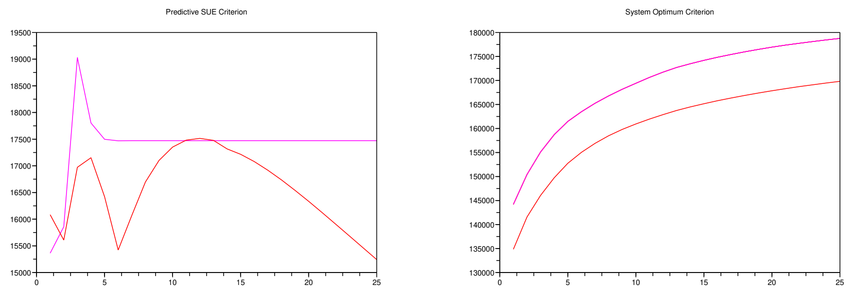

Figure 1 – *Comparison of system dynamics estimated i) with PTC,* φ *learning (purple) and ii) without PTC learning (in red). Horizontal axis:* τ *iteration (day) index*

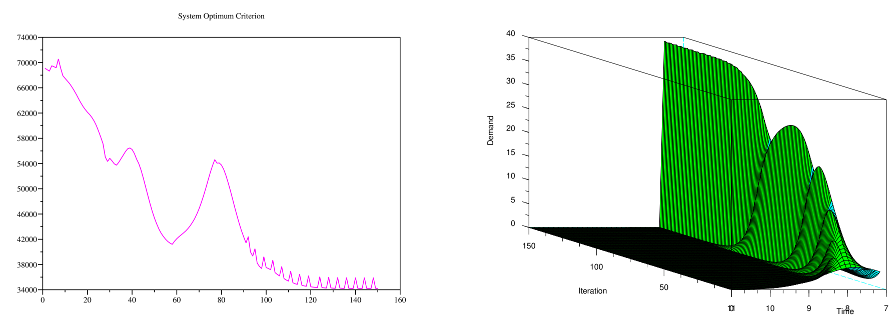

Figure 2 – *Emergence of a period 4 solution with constant smoothing coefficients. Left: SO as a function of* τ*, right departure time distribution*

(purple curves) and ii) without *PTC* learning (red curves). Two criteria have been chosen, in ordre to synthetize the system dynamics, the SO **system optimum** criterion (left) which measures the total cost spent by all travellers in the network, and the **SUE (User Equilibrium) criterion** (right) which measures the proximity with the user equilibrium for departure times. The SUE criterion is defined as

$$SUE \stackrel{def}{=} \int_{\mathcal{T}_d} dt \sum_{w \in \mathcal{W}, t_a \in \mathcal{T}_a} \left[PTC_w^{t_a}(t) - \min_{s \in \mathcal{T}_d} PTC_w^{t_a}(s) \right] .D_w^{t_a} \varphi_w^{t_a}(t)$$

The SO criterion (on the right) converges in both cases but increases (by 25%) which implies that the user equilibrium actually deteriorates the collective state of the system. The SUE criterion (left part of Figure 1) also exhibits convergence at different levels, but with oscillations in the case without PTC learning. The long term oscillation (SUE without *PTC* learning, in red) suggests oscillations of increasing period. These are actually observed in many configurations, and indicate instability of the route choice equilibrium process and unstable equilibria.

Periodic solutions are likely to occur. An example of such a situation is depicted by Figure 2, which shows the emergence of a periodic solution of period 4 (obtained with constant smoothing coefficients). The number of iterations for the emergence of this solution is all the greater than the smoothing coefficients are smaller. The emergence of the periodic solution is particularly apparent on the SO versus iteration diagram, Figure 2, left. The corresponding travel demand (versus iteration and time) is illustrated by Figure 2. Precisely we have chosen one OD and one desired arrival time, and the figure shows the corresponding distribution $\varphi_w^{\tau;t_a}(t)$ on the vertical axis. Departure time t lies on the horizontal axis, iteration (day) τ on the diagonal axis. Several major changes in the departure time pattern can be noted, before the periodic regime stabilizes. The qualitative impact of the periodic oscillations on the departure time distribution is small.

The Figure 3 shows a configuration which displays a persistent erratic behavior at all iterations. In this case no stabilization nor equilibrium formation occurs.

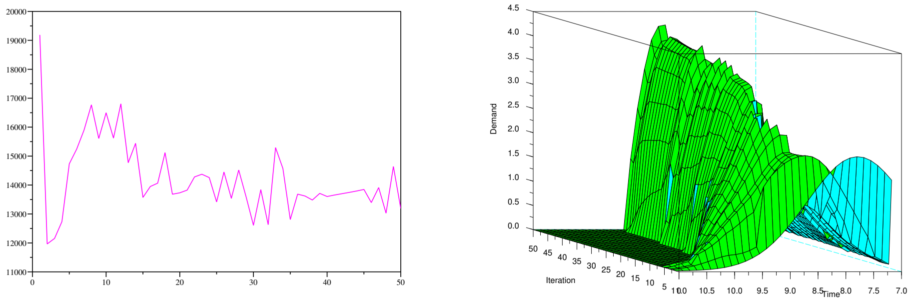

Figure 3 – *Configuration with no stabilization. The left part of the figure depicts the SO criterion, the right part the departure time distribution* $\varphi_w^{\tau,t_a}(t)$ *for one OD, one desired arrival time* ($t_a =$ 8.30 *am. Horizontal axis: departure time* t*, diagonal axis: iteration/day* τ.

On the other hand smoothing parameters chosen to satisfy the divergent series rule usually lead to the convergence of the system to an equilibrium, but whether this rule can satisfactorily represent traveller behavior is debatable.

4 DISCUSSION

The results obtained confirm and extend to the DTA context previous results described in the literature (Chiu *et al.* , 2011), Liu *et al.* (2018), Cantarella & Fiori (2022), Lebacque & Khoshyaran (2024). In particular we obtained multiple equilibria, periodic dynamics, solutions exhibiting oscillations with increasing periodicity suggestive of unstable equilibria, and dynamics exhibiting persistent instability. The nature of the medium term dynamics depend strongly on the parameters such as demand level and demand structure, but also on the learning strategies. This last point is fundamental, as it has strong implications in practice with respect to traveller information services and to transportation systems management.

References

Cantarella, Giulio Erberto, & Fiori, Chiara. 2022. Multi-vehicle assignment with elastic vehicle choice behaviour: Fixed-point, deterministic process and stochastic process models. *Transportation Research Part C: Emerging Technologies*, **134**, 103429.

Chiu, Yi-Chang, Bottom, Jon, Mahut, Michael, Paz, Alexander, Balakrishna, Ramachandran, Waller, Steven, & Hicks, Jim. 2011. Dynamic traffic assignment: A primer (transportation research circular e-c153).

Guo, Ren-Yong, & Huang, Hai-Jun. 2009. Chaos and bifurcation in dynamical evolution process of traffic assignment with flow "mutation". *Chaos, Solitons & Fractals*, **41**(3), 1150–1157.

Han, Linghui, Sun, Huijun, Wu, Jianjun, & Zhu, Chengjuan. 2011. Day-to-day evolution of the traffic network with Advanced Traveler Information System. *Chaos, Solitons & Fractals*, **44**(10), 914–919.

Iryo, Takamasa. 2019. Instability of departure time choice problem: A case with replicator dynamics. *Transportation Research Part B: Methodological*, **126**, 353–364.

Lebacque, Jean-Patrick, & Khoshyaran, Megan M. 2018. Semi-lagrangian formulation of an extended GSOM Model for Multimodal Transportation Systems. *IFAC-PapersOnLine*, **51**(9), 1–6.

Lebacque, Jean-Patrick, & Khoshyaran, Megan M. 2024. Complex behavior in day-to-day dynamics of Transportation systems. *In: FRCCS 2024 French Regional Conference on Complex Systems*. Zenodo.

Liu, Shixu, Guo, Lidan, Easa, Said M, Chen, Wensi, Yan, Hao, & Tang, Yingnuo. 2018. Chaotic behavior of traffic-flow evolution with two departure intervals in two-link transportation network. *Discrete Dynamics in Nature and Society*, **2018**, 1–11.