Line planning under crowding: A row-and-column generation approach

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1 Introduction

Public transport passengers do not only value fast and direct connections but also comfortable travel. Particularly, passengers prefer not to travel along highly crowded trajectories. However, due to the quickly growing population in metropolitan areas, public transport has become increasingly crowded. High construction costs and times make it impossible to adapt to this growth simply by extending and densifying the network. Public transport operators can, however, work towards utilizing the given rail network better by optimizing their planning. Especially changes to the *line concept*, the set of operated lines and their frequencies, can potentially alleviate crowding, as these directly influence the routes taken by passengers.

We develop a novel line planning model based on the change-and-go network of Schöbel & Scholl (2005) to trace the paths passengers take through the network. We model the impact of crowding on a passenger's perceived travel time as a linear function of the utilization of a train, resulting in a crowding term that is quadratic in the number of passengers, and inversely proportional to the frequency of that particular train. We obtain a computationally tractable mathematical model through a convex reformulation based on second-order cone constraints.

Our contributions are four-fold:

1. In view of the experience of passengers traveling in practical operations, we model the impact of crowding on the perceived passenger travel times as a quadratic term. Through a convex reformulation, we obtain a computationally tractable mixed-integer second-order cone program.

2. To solve line planning problems with crowding at scale, we develop a scalable row-and-column generation algorithm. We separate the convex cone constraints and lower-bound our solution with resulting linear cuts and generate additional passenger routes on the fly. We employ a novel diving heuristic to obtain integer solutions. Compared to Gurobi, we obtain near-optimal solutions in a fraction of the computation time.

3. We show that neglecting crowding effects in line planning results in suboptimal line plans, both on artificial grid instances commonly used in the line planning literature and on the Beijing metro network. While some parts of the network are highly crowded, others see very few passengers. Even a low crowding penalty obtains high-quality line plans.

4. To evaluate the impact of passengers selecting routes, we evaluate the line plan against travel-time minimizing passengers using the Wardrop principle. We find that the gap between the social- and user-optimal routing is very small, indicating that the line plan is good.

2 Methodology

We consider the line planning problem under congestion which we define on a change-and-go network (see, e.g., Schöbel & Scholl, 2005), a graph consisting of nodes referring to station-line pairs and arcs referring to direct connections of lines, waiting at stations, and transferring at stations. For brevity, we omit further details here.

For a given line pool $\mathcal{L}$, a candidate sets of lines that can be built, the operator determines which lines to operate at which frequency. The operator minimizes the passengers' perceived travel times, which comprise physical travel costs and a crowding penalty. Per passenger, this penalty increases linearly in the number of other passengers traversing this arc, resulting in a quadratic penalty per arc. We omit further physical and operational constraints to obtain a clear view of the crowding effect, except for a budget of B.

After the line plan has been established, passengers are assigned a route from their origin to their destination to minimize the average perceived travel time. All passengers can use any feasible route through the network, precomputed as route set $\mathcal{R}$.

2.1 Reformulated mixed-integer second-order cone program

The line planning problem under congestion can be modeled as a program with linear constraints and a non-linear objective

$$\min \sum_{a \in \mathcal{A}} c_a x_a + \sum_{a \in \mathcal{A}: y_{\ell(a)} > 0} \frac{\gamma_a x_a^2}{y_{\ell(a)}} \tag{1}$$

where c_a and γ_a are cost factors, x_a is an arc flow, and $y_{\ell(a)}$ is the frequency of the line this arc belongs to. This non-linear program cannot be solved by standard methods. We thus cast the program into a mixed-integer second-order cone program with the objective (2).

$$\begin{aligned} \min & \sum_{a \in \mathcal{A}} c_a x_a + \sum_{a \in \mathcal{A}} \Theta_a && && (2) \\ \Theta_a y_{\ell(a)} & \geq \gamma_a x_a^2 && \forall a \in \mathcal{A}, && (3) \\ & \dots \end{aligned}$$

The rotated second-order cone constraints (3) ensure that the flow is 0 if the frequency y_a is 0, and ensure that $\Theta_a \geq \gamma_a x_a^2 / y_{\ell(a)}$ otherwise.

2.2 Cutting planes for SOCP constraints

While most contemporary commercial solvers can handle smaller mixed-integer second-order cone programs, we handle the second-order cone constraints using a cutting plane method to solve larger instances. We prove that adding cuts

$$\frac{\Theta_a}{\gamma_a} \geq \frac{2\hat{x}_a}{\tilde{y}_{\ell(a)}} x_a - \frac{\hat{x}_a^2}{\tilde{y}_{\ell(a)}^2} y_{\ell(a)}, \tag{4}$$

instead of the non-linear constraints (3) results in optimal solutions where $\tilde{y}_{\ell(a)} = \max\left\{\hat{y}_{\ell(a)}, \varepsilon_1\right\}$ for a small $\varepsilon_1 > 0$ to deal with zeros and avoid numerical issues.

2.3 Column generation

The second challenge for solving the reformulated mixed-integer second-order cone program lies in the precomputed set of routes necessary to compute the costs per passenger. However, we can efficiently handle this challenge using column generation. Starting from a limited set of routes, we solve the restricted master problem using Farkas pricing (Kowalczyk & Leus, 2018) to restore feasibility if budget constraints and currently available routes per passenger contradict each other. Given the solution, we find the shortest route for each passenger in a directed acyclic graph. If the shortest route reduces costs, we add the associated column.

To obtain integer-feasible solutions, we encapsulate the row-and-column generation in a diving heuristic which fixes lines running at a high frequency without harming feasibility.

3 Results

We test the performance of the algorithm as well as the benefit of considering crowding during line planning on two sets of instances, a 5x5-grid detailed in Friedrich *et al.* (2017) and the Beijing metro network, which is one of the world's largest networks and currently consists of 277 stations and 19 lines.

Consistently across various parameter settings for budget and crowding factors, the row-and-column generation procedures obtain comparable solutions to Gurobi in a fraction of time and consider only a small subset of routes. Among the row-and-column generation procedures, we find that first generating cuts and then columns is faster than other algorithmic setups.

We then investigate how considering congestion in the decision-making affects passengers' perceived and actual travel times, as well as the line plans.

Insight 1 *Incorporating crowding effects in line planning reduces the average perceived travel time of passengers, even when the crowding factor is small. However, this comes at the expense of an increase in average travel time.*

Figure 1 shows the cumulative distribution of congestion levels for different crowding factors. The curves keep shifting to the left as the factor increases, showing decreased crowding. This effect can be explained by operators adapting their line plan when they pay attention to crowding and passengers' objectives. We observe that even a small crowding factor has an impact since their resulting line plans are often similar. However, minimizing the average perceived travel times, some passengers are rerouted across routes that are not even among their fastest ten routes (according to physical travel time) to circumvent congested line segments.

As a consequence of this rerouting, the average physical travel time increases:

Insight 2 *By incorporating crowding effects in line planning, more passengers make transfers, and as a result, passengers' average physical travel times increase by up to 4.48%.*

We observe that shorter perceived travel times do not only occur through passenger rerouting but are frequently the result of fundamentally different line plans:

Insight 3 *Incorporating crowding effects results in structurally different line plans with fewer lines with higher frequencies.*

Figure 2 shows the line plan without (left) and with (right) crowding. Public transport operators might be able to use this as a rule of thumb for generating passenger-friendly line plans.

Since passengers do not always resort to the recommended route through a network but instead minimize their perceived travel time, we also compare it to the user equilibrium. We evaluate the user equilibrium for a given line plan using a convex quadratic model that exceeds the space in this extended abstract.

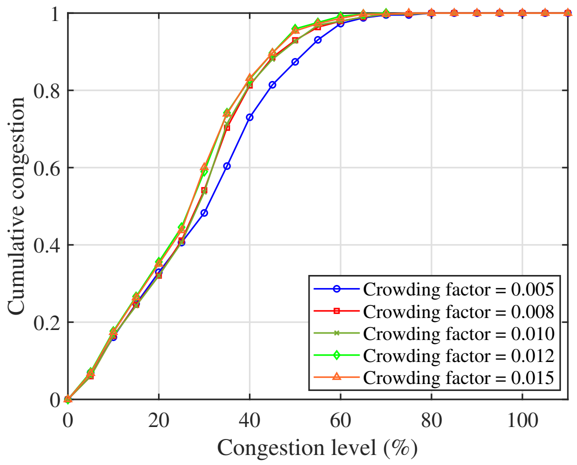
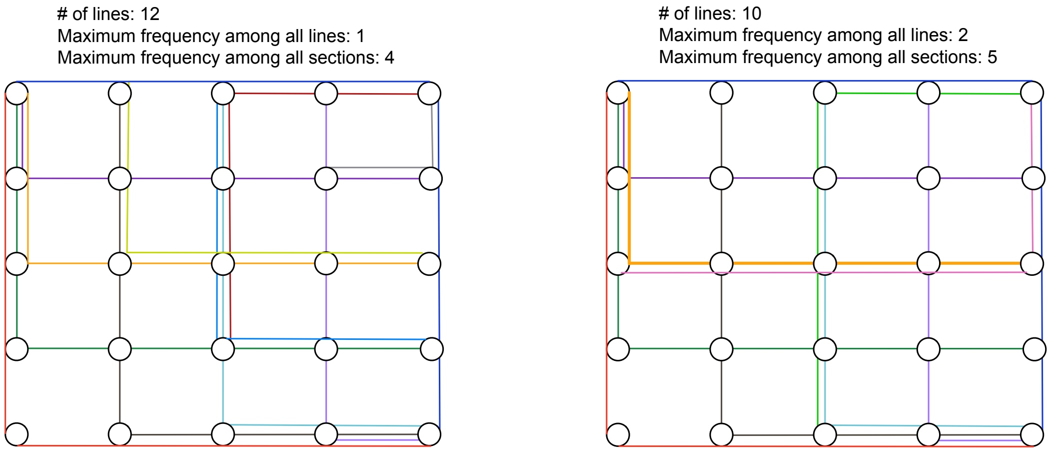


Figure 1 – *The cumulative distribution of congestion levels under optimized line plans with various crowding factors.*



(a) *Crowding factor = 0*

(b) *Crowding factor = 0.20*

Figure 2 – *Optimal line plans without and with incorporating crowding effects*

Insight 4 *The system-optimal routing deviates only slightly from the user equilibrium for varying crowding factors and budgets.*

Among all considered instances, the maximum deviation in perceived travel time is 0.01%. As such, we conclude that integrating the passengers' self-interested routing is not necessary. If some instances show larger deviations, user equilibria can be derived consecutively using a cutting-plane approach.

References

Friedrich, Markus, Hartl, Maximilian, Schiewe, Alexander, & Schöbel, Anita. 2017. Integrating Passengers' Assignment in Cost-Optimal Line Planning. *Pages 5:1–5:16 of: 17th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2017).* Open Access Series in Informatics (OASIcs), vol. 59. Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

Kowalczyk, Daniel, & Leus, Roel. 2018. A Branch-and-Price Algorithm for Parallel Machine Scheduling Using ZDDs and Generic Branching. *INFORMS Journal on Computing*, **30**(4), 768–782.

Schöbel, Anita, & Scholl, Susanne. 2005. Line planning with minimal transfers. *In: Proceeding of 5th Workshop on Algorithmic Methods and Models for Optimization of Railways.*