Budget-constrained user equilibrium: A quasi-variational inequality approach

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1 INTRODUCTION

User equilibrium is a key concept in transportation modeling for managing transportation networks. Traditional models assume travelers can choose any route between their origin and destination. However, in reality, each route involves resource consumption, such as time or energy, and is influenced by factors like congestion. Travelers must select routes where resource consumption stays within their budget. While some studies have considered such budget constraints with flow-independent resource consumption (Jiang *et al.*, 2012), actual consumption is often flow-dependent.

This paper explores budget-constrained user equilibrium with flow-dependent resource consumption using a general setting for congestion games involving users and facilities. Users decide whether to visit a facility to gain utility, consuming part of their budget. Both utility and resource consumption depend on aggregate demand. Resource consumption has an upper limit, preventing strategies that exceed the budget. This model applies to scenarios such as facility planning in entertainment parks, where users face queuing and limited time, and energy-constrained transport tools like electric or unmanned aerial vehicles, where congestion impacts battery consumption.

The literature has scarcely addressed this type of user equilibrium, with most related studies focusing on electric vehicle service planning. He *et al.* (2014) first formulated two network equilibrium models considering flow-independent and flow-dependent battery consumption respectively. Xu *et al.* (2017) expanded on this by incorporating both battery electric and gasoline vehicles. However, these works primarily focus on formulating equilibrium conditions and lack efficient algorithms tailored to this problem. Liu & Song (2018) proposed a solution approach by transforming the equilibrium into a non-convex optimization problem. Further exploration of the problem's structure is needed to develop more efficient algorithms. Niroumand *et al.* (2022) employed a penalty-based algorithm that iteratively adjusts penalties on each sub-path. However, this penalty scheme is somewhat heuristic and lacks a theoretical guarantee.

In this study, we introduce a novel application of Quasi-Variational Inequality (QVI) to formulate the general budget-constrained user equilibrium. We establish the equivalence between the equilibrium conditions and the QVI problem. To solve it, we employ an Augmented Lagrangian Method (ALM)-like algorithm. This approach addresses a series of Variational Inequalities (VIs), within which we solve the restricted master problem and generate new feasible strategies through column generation (CG). Finally, we demonstrate the effectiveness of our algorithm through numerical experiments.

2 THE MODEL AND ALGORITHM

2.1 Model setup

We consider a scenario with I types of users, each having a total demand λ_i, and J facilities, each with a service rate μ_j. Each user has a strategy set defined as:

$$\mathcal{S}_i \subseteq \mathcal{S} = \{0,1\}^J, \tag{1}$$

indicating that users can choose whether to visit any facility j to gain a benefit ω_{ij} while incurring a waiting time cost c_{ij}. We use $\delta_{sj} = 1$ to indicate that strategy $s \in \mathcal{S}$ includes choosing facility j and 0 otherwise. The proportion of type-i users choosing strategy s is represented by r_{is}.

The waiting time at facility j, $t_j(q_j)$, is a continuous, strictly increasing function of the aggregate demand rate q_j from all user types to the facility, calculated as:

$$q_j = \sum_{i\in[I]} \lambda_i \sum_{s\in\mathcal{S}_i} r_{is}\delta_{sj}. \tag{2}$$

Similarly, the waiting cost $c_{ij}(q_j)$ is also a continuous, strictly increasing function of q_j. We calculate the utility as the difference between the benefits and the costs incurred at the chosen facilities, which is then continuous and strictly decreasing with q_j, expressed as:

$$w_{is} = \sum_{j\in[J]} \left(\omega_{ij} - c_{ij}\right)\delta_{sj}. \tag{3}$$

Each user type i has a time budget τ_i, imposing a constraint on the total waiting time they can afford. This constraint is expressed as:

$$e_s = \sum_{j\in[J]} t_j\delta_{sj} \le \tau_i. \tag{4}$$

At the user equilibrium, the following two conditions should be held:

$$\text{If } e_s > \tau_i,\ s\in\mathcal{S}_i,\ i\in[I], \text{ then } r_{is} = 0; \tag{5a}$$

$$\text{If } r_{is_1} > 0 \text{ and } e_{s_2} < \tau_i,\ s_1, s_2 \in\mathcal{S}_i,\ i\in[I], \text{ then } w_{is_1} \ge w_{is_2}. \tag{5b}$$

The first condition indicates that a strategy can only be selected if the budget it consumes do not exceed the upper limit. The second condition implies that for a chosen strategy, the utility it generates should be at least as much as any other feasible strategy.

2.2 A QVI formulation

We address a QVI problem, denoted as QVI($\boldsymbol{\Omega}, \boldsymbol{F}$), involving a vector-valued function $\boldsymbol{F}$ and a set-valued function $\boldsymbol{\Omega}$. The objective is to find a vector $\boldsymbol{r}^* \in \boldsymbol{\Omega}(\boldsymbol{r}^*)$ such that:

$$(\boldsymbol{r} - \boldsymbol{r}^*)^\top \boldsymbol{F}(\boldsymbol{r}^*) \le 0,\ \forall \boldsymbol{r} \in \boldsymbol{\Omega}(\boldsymbol{r}^*). \tag{6}$$

In our context, $\boldsymbol{r}$ represents a vector with components r_{is}, and $\boldsymbol{F}$ is defined component-wise as:

$$\boldsymbol{F}(\boldsymbol{r}) = \left[\lambda_i \cdot w_{is}(\boldsymbol{r})\right]_{s\in\mathcal{S}_i,\ i\in[I]}. \tag{7}$$

The set $\boldsymbol{\Omega}(\boldsymbol{r}^*)$ is characterized by the following conditions:

$$\boldsymbol{\Omega}(\boldsymbol{r}^*) = \left\{\boldsymbol{r}\middle|\boldsymbol{1}^\top\boldsymbol{r}_i = 1,\ \boldsymbol{r}_i \ge 0,\ \forall i\in[I];\ 0 \le r_{is} \le \gamma_{is}(\boldsymbol{r}^*),\ \forall s\in\mathcal{S}_i,\ i\in[I]\right\}, \tag{8}$$

where γ_{is} represents the maximum proportion of i-type users that can utilize strategy s assuming other users do not alter their choices. It is calculated by:

$$\gamma_{is}(\boldsymbol{r}^*) = \mathrm{Proj}_{[0,1]}\left(\max\left\{r_{is} | \tau_i \ge e_s(\tilde{\boldsymbol{r}}_{is}(\boldsymbol{r}^*, r_{is}))\right\}\right), \tag{9}$$

where $\tilde{\boldsymbol{r}}_{is}(\boldsymbol{r}^*, r)$ is the vector obtained by replacing the (i,s)-th entry with r while keeping all other entries the same as in $\boldsymbol{r}^*$.

Proposition 1 *A vector* $\boldsymbol{r}^*$ *satisfies the user equilibrium conditions* (5) *if and only if it solves the* QVI$(\boldsymbol{\Omega}, \boldsymbol{F})$ *problem.*

We will discuss existence and uniqueness of the user equilibrium in the full paper.

2.3 Solution algorithms

Algorithms intended for general QVI problems can be tailored to solve (6). We pick the algorithm proposed by Pang & Fukushima (2005), which employs an ALM-like strategy to address flow-dependent constraints by using the augmented Lagrangian and solving a series of VIs. The solution procedures bear similarity with the heuristic method of Niroumand *et al.* (2022), yet it is further refined within the QVI and ALM framework. The subsequent Algorithm 1 details our implementation based on the approach of Pang & Fukushima (2005).

Algorithm 1 An ALM-like algorithm for QVI$(\boldsymbol{\Omega}, \boldsymbol{F})$

Initialize: $\varrho^0 \leftarrow 1$, $\varepsilon_1^0 \leftarrow +\infty$, $\varepsilon_2^0 \leftarrow +\infty$, $u_{is}^0 \leftarrow 0$, $\forall s \in \mathcal{S}_i,\ i \in [I]$, and $k \leftarrow 0$

Define:

$$\bar{\boldsymbol{\Omega}} = \left\{ \boldsymbol{r} \middle| \mathbf{1}^\top \boldsymbol{r}_i = 1,\ \boldsymbol{r}_i \geq 0,\ \forall i \in [I] \right\} \tag{10}$$

while $\varepsilon_1^k > \bar{\epsilon}_1$ or $\varepsilon_2^k > \bar{\epsilon}_2$ **do**

Define:

$$\boldsymbol{F}^k(\boldsymbol{r}) = \boldsymbol{F}(\boldsymbol{r}) - \left[\left(u_{is}^k + \varrho^k \left(r_{is} - \gamma_{is}(\boldsymbol{r}) \right) \right)^+ \right]_{s \in \mathcal{S}_i, i \in [I]} \tag{11}$$

Solve: $\boldsymbol{r}^k \leftarrow \mathrm{VI}(\bar{\boldsymbol{\Omega}}, \boldsymbol{F}^k)$

Solve: $w_i \leftarrow \max_{x_j \in \{0,1\}} \sum_{j \in [J]} w_{ij}(\boldsymbol{r}^k) x_j$ s.t. $\sum_{j \in [J]} t_j(\boldsymbol{r}^k) x_j < \tau_i,\ \forall i \in [I]$

Update:

1. $\varrho^{k+1} \leftarrow \rho \cdot \varrho^k$
2. $\varepsilon_1^{k+1} \leftarrow \frac{\sum_{i \in [I]} \lambda_i \sum_{s \in \mathcal{S}_i} r_{is}^k (e_s - \tau_i)^+}{\sum_{i \in [I]} \lambda_i \tau_i}$
3. $\varepsilon_2^{k+1} \leftarrow \frac{\sum_{i \in [I]} \lambda_i \sum_{s \in \mathcal{S}_i} r_{is}^k (w_i - w_{is})^+}{\sum_{i \in [I]} \lambda_i w_i}$
4. $u_{is}^{k+1} \leftarrow \left(u_{is}^k + \varrho^k \left[r_{is} - \gamma_{is}(\boldsymbol{r}) \right] \right)^+,\ \forall s \in \mathcal{S}_i,\ i \in [I]$
5. $k \leftarrow k + 1$

end while

The algorithm iteratively updates the function $\boldsymbol{F}^k$ by incorporating penalty terms and dual variables, and then solves VI$(\bar{\boldsymbol{\Omega}}, \boldsymbol{F}^k)$ to find a vector $\boldsymbol{r}^* \in \bar{\boldsymbol{\Omega}}$ such that:

$$\left(\boldsymbol{r} - \boldsymbol{r}^* \right)^\top \boldsymbol{F}^k(\boldsymbol{r}^*) \leq 0,\ \forall \boldsymbol{r} \in \bar{\boldsymbol{\Omega}}. \tag{12}$$

During each iteration, the penalty parameter ϱ^k is increased, and the error measures along with dual variables are updated to reflect the feasibility of $\boldsymbol{r}^k$. Regarding the error measures, we propose the following:

Proposition 2 *If* $\varepsilon_1^{k+1} = 0$*, then* $\boldsymbol{r}^k$ *satisfies the equilibrium condition* (5a)*. If* $\varepsilon_2^{k+1} = 0$*, then* $\boldsymbol{r}^k$ *satisfies the equilibrium condition* (5b)*.*

To address VI$(\bar{\boldsymbol{\Omega}}, \boldsymbol{F}^k)$, we employ the projection-type algorithm (Harker & Pang, 1990) with a CG framework (Lawphongpanich & Hearn, 1984).

We will also discuss the convergence results of Algorithm 1 in the full paper.

3 NUMERICAL RESULTS

We consider a scenario with $I = 10$ types of users and $J = 20$ facilities. The strategy set is defined as $\mathcal{S}_i = \{0, 1\}^J$. The demand λ_i is set between $4,000$ and $6,000$, the service rate μ_j is set between 50 and 450, the benefit ω_{ij} is set between 125 and 375, and the budget τ_i is set between 500 and $1,500$. The waiting time is calculated as $t_j(q_j) = \frac{q_j}{\mu_j}$, and the waiting cost is proportional to the waiting time with the value of time set to 1. Figure 1 (left) displays the error measures ε_1^{k+1} and ε_2^{k+1}. Figure 1 (right) present the waiting time $t_j(q_j(\boldsymbol{r}^k))$ at each facility, with the red line representing the average waiting time across all facilities. These visualizations both demonstrate the algorithm's convergence.

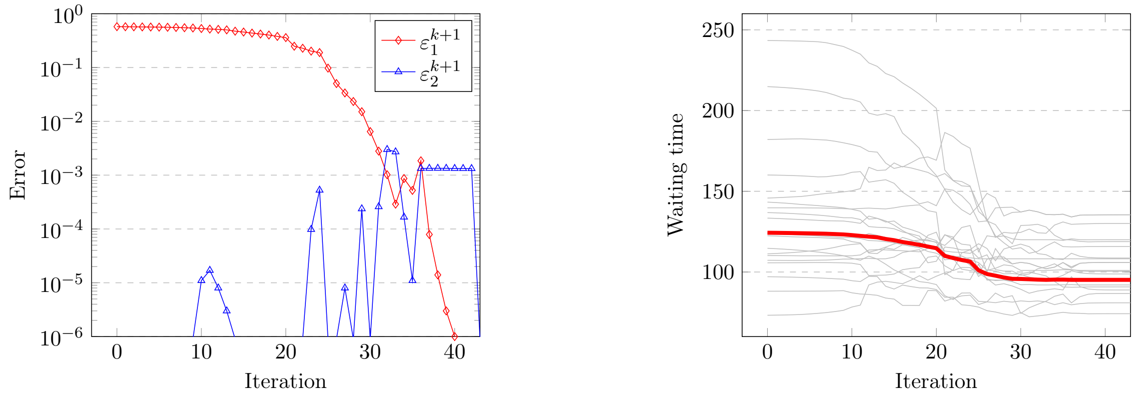

Figure 1 – *Error measures and queue waiting time*

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