The theoretical role of the pure transfer penalty when determining whether to split a public transport line

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1 INTRODUCTION

Consider a public transport system serving a corridor. There is a long history of analytical models to determine the optimal strategic decisions, such as frequencies, stops spacing, or subsidies (Mohring, 1972, Jara-Díaz & Gschwender, 2003, Fielbaum, 2024, Coulombel & Monchambert, 2023). However, little has been done to understand the network-related aspects. In particular, two usual networks found on different cities worldwide is either having a line covering the whole corridor, or two lines each serving one segment. The latter is typically observed with the two lines converging at the CBD, or in a feeder-trunk manner connected at a subcenter. In this paper, we investigate theoretically under which conditions a divided line is better than a single one, minimizing the sum of users' and operators' costs. In particular, as having two lines would require transfers, we focus on the role of the *pure transfer penalty* PTP, a figure representing how much do users penalize the very fact of interrupting their journey, and for which a wide range of values have been reported in the literature (Yap *et al.*, 2024, Garcia-Martinez *et al.*, 2018).

2 FORMULATION AND METHODOLOGY

Similar to previous analytical literature on the topic, we consider a linear model, where we focus on the supply aspects by considering a given demand pattern. We define the General Linear City (Figure 1) as a path of k nodes $\{n_1, n_2, ..., n_k\}$, where all passengers travel in the same direction. We denote by $y_{g,h}$ the demand from node g to node h and the distance is measured in travel time. We denote the flow crossing across arc $u, u+1$ as Eq. 1.

$$\bar{y}_u = \sum_{g \le u \, u+1 \le h} y_{g,h} \qquad (1)$$

$$l_0 = [n_1, n_2, ..., n_k] \qquad (2)$$

$$l_{(0,i)} = [n_1, n_2, ..., n_i] \qquad (3)$$

$$l_{(i,k)} = [n_i, n_{i+1}, ..., n_k] \qquad (4)$$

$$S_0 = \{l_0\} \qquad (5)$$

$$S_i = \{l_{(0,i)}, l_{(i,k)}\} \qquad (6)$$

We define a line by means of its sequence of stops (Eqs. 2 - 4). The first one is the complete line in the General Linear City (it stops in each node), and the other two are the result of dividing the line in the i-th node, which enables offering different frequencies and bus capacities at each segment. We define the line structure S_0 (Eq. 5) that contains the complete line, and the alternative line structure S_i (eq. 6) formed by the two divided lines divided at node i.

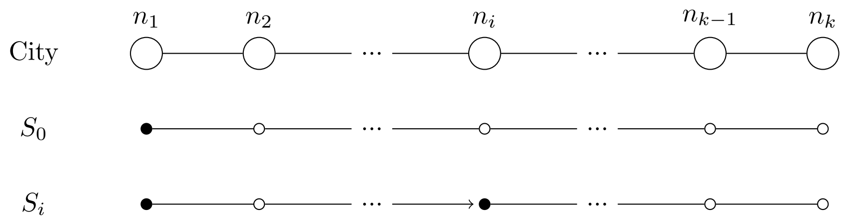


Figure 1 – *Linear city layout and line structures.*

The objective is to minimize the total value of the resources consumed (VRC), i.e. the sum of operators' and users' cost. The design variables are the line structure S (S_0 or one of the S_i), the frequencies f_l of all the lines l involved in S, and the vehicles' capacity of every line K_l. The VRC is given by:

$$VRC(S,f,K) = \sum_{l \in S} B_l(S,f)(c_0 + c_1 K_l(S,f)) + Y(p_{\text{iv}}\bar{t}_{\text{iv}}(S,f) + p_w \bar{t}_w(S,f)) + p_R R(S,f) \quad (7)$$

Where $B_l(S,f)$ is the fleet size. c_0 and c_1 are, respectively, the fixed cost per vehicle and the component that grows linearly with its size. $\bar{t}_{\text{iv}}(S,f)$ is the average in-vehicle travel time and p_{tr} is its value; $\bar{t}_{\text{w}}(S,f)$ is the average waiting time with value of waiting p_w, and $R(S,f)$ the total number of transfers, each with a penalty p_R. Note that p_R is independent of the additional waiting time, so it represents only the annoyance caused by trip interruption. p_R is typically measured in equivalent in-vehicle minutes EIVM, with reported values ranging from 4 to 18 EIVM. It is convenient to split a line at node i iff $\min_{f_0,K_0} VRC(S_0,f_0,K_0) > \min_{f,K} VRC(S_1,f,K)$. A careful analysis of Eq. (7) yields the following theorem.

Theorem 1: Consider the linear city network $\{n1, n2, ..., nk\}$, where the passenger distribution is given by the upper triangular matrix y and the line structures are $S_0 = \{l_0\}$ and $S_i = l_{(0,i)}, l_{(i,k)}$ with $l_0 = [n_1, n_2, ..., n_k]$, $l_{(0,i)} = [n_1, n_2, ..., n_i]$ and $l_{(i,k)} = [n_i, n_{i+1}, ..., n_k]$. Then it is convenient to split at node i if and only if

$$\begin{aligned} &4Tc_0\left((k-1)f_0^* - (i-1)f_{(0,i)}^* - (k-i)f_{(i,k)}^*\right) - \tau_i(2tc_0 + p_R) \\ &+2Tc_1\left((k-1)\max_{u\in[k-1]}\bar{y}_u - (i-1)\max_{u\in[i-1]}\bar{y}_u - (k-i)\max_{u\in[k-1]\setminus[i-1]}\bar{y}_u\right) > 0 \end{aligned} \quad (8)$$

From Eq. 8, the three conditions than favor the division can be inferred, and are shown in Table 1. Note that the first condition reveals that transfers have two effects: their direct penalty through the PTP, and the additional waiting and boarding-and-alighting times.

3 The Divisibility Index: Definition and Application

Conditions in Table 1 can be summarized in the *Divisibility Index* (DI) to express how favorable is a node to the division.

Definition 1: Denote $\zeta_l(n)$ as the number of passengers crossing the n-th node, and l_i the length of the section with the largest flow. Then, the divisibility index (DI) of node i is:

$$DI(i) := \left(1 - \frac{\tau_l(i)}{Y}d_1\right)\left|\max_{1\le n\le i}\zeta_l(n) - \max_{i\le n\le k}\zeta_l(n)\right|\left(1 + \left(|l_i| + \frac{|l_i|}{|l|}\right)d_2\right) \quad (9)$$

Table 1 – *Conditions favoring division.*

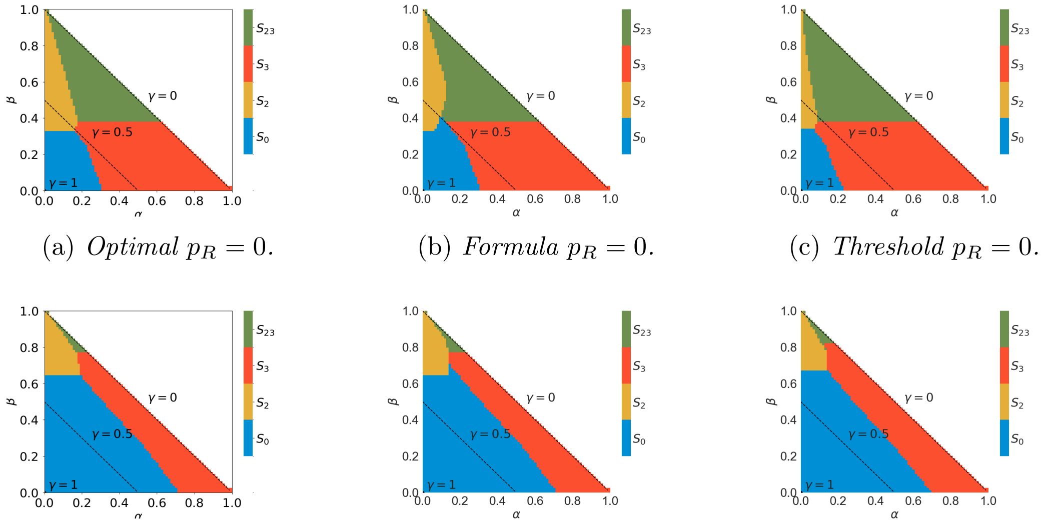
	Condition	Interpretation
1	Low number of transfers in i	Transfers increase boarding and alighting time (and the waiting time), and are penalized.
2	Large difference in peak flow between $[0, i]$ and $[i, k]$	Savings thanks to the fleet reduction in the section with the lowest flow.
3	Longer section length ($[0, i]$ or $[i, k]$) of lower peak flow	Savings thanks to the idle capacity reduction in the section with the lowest flow.

The DI can be used to develop simple rules to decide whether and where to split a line. First, we consider the best candidate for division as the node with the largest DI. The question then is whether splitting the line at that node does improve the system. Evidently, this question can be answered by the evaluation of Eq. 8 at the candidate node; if positive, then divide the line: we call this the *formula procedure*. A faster alternative is determining if the DI is large enough, compared to a predetermined threshold appropriately chosen. If the DI of the best candidate is above the threshold, then split the line at the node, otherwise, don't: the *threshold procedure*. In both cases, it is natural to generalize the procedure to the possibility of admitting more than one division per line in an algorithm; it starts with a complete line and determines whether or not there is a split at its node with the greatest DI. If the decision is to split, then the process continues recursively on both obtained segments.

The division algorithms are tested (and the relevance of the PTP is examined) in a particular case of the linear city inspired by the parametric city mode (PCM, Fielbaum *et al.* (2016)) that considers n zones, each with a periphery and a subcenter, plus a CBD.. The linear city version includes four nodes only representing Periphery-own Subcenter-CBD-Distant subcenter, i.e., a one-zone version of the PCM. The demand distribution is commanded by parameters α, β, γ, that can be interpreted respectively as how monocentric, polycentric, or dispersed the city is, with $\alpha + \beta + \gamma = 1$. Crucially, in this city we can evaluate all the possible lines structures and find the optimal one, providing an exact benchmark to compare our methods.

Following the formulation notation, S_0 is the complete line structure, S_2 is the line structure divided in the second node, S_3 is divided in the third node and S_{23} has a simultaneous division in the second and third node. As inferred above, in the α vs β space it is possible to simulate monocentric, polycentric and dispersed the cities. When α and β are low, the main destination in the Distant subcenter, so there are no conditions (Table 1) to the division. If α is high, then the CBD is the main destination, so the conditions 1 and 2 accomplish for a division in the CBD. If β is high, the main destination is the Subcenter, so all the condition accomplish for a division in the Subcenter. Otherwise if α and β are both high, the conditions accomplish for a division in the Subcenter and the CBD. So, we hope S_0 to be optimal close to he origin, S_2 optimal for high values of β, S_3 optimal for high values of α and S_{23} optimal when α and β are high enough.

In the Figure 2 there is a numerical example with different p_R values, where the DI parameters are $d_1 = 0.9$ and $d_2 = 0.08$. The optimal result (Figures 2a and 2d) follows the scheme described above. More on, we can observe that when the transfers penalty p_R increase, then the dominance of the complete line structure noticeably increase and the double division line structure almost disappear. On the other hand, if we compare the algorithms results (Figures 2b, 2c, 2e and 2f) with the optimal figures, is clear the successful of the complete line divisions. In 2c the threshold is set in 0.15, but in 2f is set in 1.78. So, we can conclude that the pure transfer penalty definitely has a role. However, the DI also shows that there are other important conditions, and that the impact of transfers is not limited to the pure transfer penalty value.



(a) *Optimal* $p_R = 0$. (b) *Formula* $p_R = 0$. (c) *Threshold* $p_R = 0$.

(d) *Optimal* $p_R = 0.73$. (e) *Formula* $p_R = 0.73$. (f) *Threshold* $p_R = 0.73$.

Figure 2 – *Result example,* p_R *measured in EIVM.*

4 DISCUSSION

We have analytically established the conditions determining whether a divided line is better than a single one: a low number of transfers, a significant difference in peak flow on each side of the division, and a longer section of lower peak flow. These three conditions are synthesized into a new divisibility index DI to measure the suitability for division at any a given node. Two algorithms that use the divisibility index are proposed to determine whether and where to split a line. An application of these algorithms to the so-called parametric linear city proves that they provide an excellent approximation of the true optimal network. Although our results are obtained within the context of a corridor, they are not limited to it. The expansion of the algorithms using the DI to the general parametric city model and to any network with a more complex topology is indeed the next step in this research line, using the division algorithms to enhance traditional heuristics in any network.

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