Error bounds for stochastic user equilibrium traffic assignment

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1 INTRODUCTION

In stochastic user equilibrium traffic assignment, we derive bounds on the distance between a given feasible solution and the equilibrium solution. Traditional gap functions, such as the difference between the average experienced travel time and the shortest path travel time, are used to measure how well the current solution satisfies the equilibrium principle. However, these functions do not directly quantify how close the traffic flows are to equilibrium flows. Researchers have investigated this question numerically, as in the studies by Boyce et al. [2004] and Patil et al. [2021], showing how specific numerical values of a gap function correlate with convergence of flows and other aggregate metrics (such as total system travel time and vehicles miles travelled). The present goal is to derive mathematical bounds to complement the extant numerical evidence. To this end, we use stochastic user equilibrium (SUE) model, assuming logit path choice with dispersion parameter θ. Even though the above studies used deterministic user equilibrium (DUE), a major advantage of SUE for our purposes is that the "target" solution is continuous in the current solution.

The intent of our study is to provide guidance on early-termination criteria to reduce run times. This is important because traffic assignment problem is often a subproblem to a more complex bilevel optimization and must be solved many times. Our approach is based on Taylor's theorem, as applied to the fixed-point formulation of the stochastic user equilibrium assignment, and provides upper bounds on differences in both aggregate metrics (total travel time, distance traveled) and disaggregate metrics (link flows, path flows). We test our bounds on real world networks showing the practical applications and tightness of these bounds in practice.

2 METHODS

In this section, we outline the approach for deriving an upper bound on the distance of any feasible path flow from the equilibrium flow. The upper bounds on link flows and other aggregate metrics follow directly. We first define some notation and present our main result, followed by a proof sketch (the full paper will contain all of the details). For any origin-destination (OD) pair, let h represent a feasible path flow, and let H denote the set of feasible path flows. In logit SUE, path costs determine the probability of choosing a particular path. We define $\mathcal{P}(c)$ as a function that takes a vector of path costs c and returns the probability distribution over the paths. The expected flow on each path depends on the path costs and is given by the function $\mathcal{H}(c)$, which produces a vector of path flows h based on the cost vector c. We refer to $\mathcal{H}(c)$ as the "target" flows because it tells us what the path flows should be for a given vector of path costs. The path costs themselves are calculated through network loading, in which the path flows determine the link flows, which, in turn, influence the path costs. To represent this relationship, we define $\mathcal{C}(h)$, a function that returns the vector of path costs corresponding to a given path flow vector h.

The stochastic user equilibrium can be formulated as a fixed-point problem: find $\hat{h} \in H$ such that $\hat{h} = \mathcal{H} \circ \mathcal{C}(\hat{h})$. It is well-established that, under mild conditions, such an equilibrium exists and is unique. Moreover, this equilibrium can be expressed as the minimum point of a strictly convex function [Boyles et al., 2020]. Crucially, the function $\mathcal{H}$ is continuous, which contrasts with the deterministic user equilibrium (DUE), where small changes in path costs in the corresponding mapping (the "all-or-nothing assignment") can cause significant shifts in target flows.

We denote the composite function $\mathcal{H} \circ \mathcal{C}$ by ψ, referring to it as the logit mapping function. The Jacobian of this function evaluated at a given flow h is denoted $\psi'(h)$. To proceed carefully, we re-specify the mapping $\mathcal{H}$ in a more general form that allows for the variation of h without changing the demand d, and we derive its Jacobian. By applying the chain rule, the Jacobian of $\psi = \mathcal{H}(h, \mathcal{P}(\mathcal{C}(h)))$ with respect to h can be expressed as:

$$\psi'(h) = \mathcal{H}'(h) + \mathcal{H}'(p) \cdot \mathcal{P}'(c) \cdot \mathcal{C}'(h). \tag{1}$$

As a preliminary result, we prove that the product of the Jacobian $\mathcal{H}'(h)$ and the difference between any two feasible path flows within the same OD pair is zero. Next, we define $\phi'(h) = \mathcal{H}'(p) \cdot \mathcal{P}'(c) \cdot \mathcal{C}'(h)$. By expanding the Jacobian, we demonstrate that if the cost functions have bounded derivatives in the feasible set of path flows, then the function $\phi'(h)$ is bounded. Specifically, there exist constants $\underline{M_1}$ and $\overline{M_1}$ such that:

$$d\theta \underline{M_1} \leq \phi'_{ij}(h) \leq d\theta \overline{M_1}. \tag{2}$$

Furthermore, assuming the cost functions are thrice continuously differentiable (i.e., C^3) and holomorphic, we show that the norm of the second and third total derivatives of the Jacobian of the logit mapping is also bounded. i.e, there exist constants M_2 and M_3 such that:

$$\|\phi^{(2)}(h)\| \leq M_3 \quad \text{and} \quad \|\phi^{(3)}(h)\| \leq M_3. \tag{3}$$

The proofs follow from the boundedness of the Jacobian, the compactness of the domain of feasible path flows, and the holomorphic nature of the link performance functions. We compute the constants M_2 and M_3 using Cauchy's estimates. With these preliminary results established, we can now proceed to prove the main results of our study.

Theorem 1 (Local Bound)**.** *For any $1 > r \geq 0$, there exists an ϵ_h such that for all path flows close to equilibrium ($\|h - \hat{h}\| \leq \epsilon_h$) the norm of difference of current path flows from equilibrium* *path flows ($\|\hat{h} - h\|$) is upper bounded by a factor that solely depends on current path flows, i.e* $\|(\hat{h} - h)\| \leq M_{hr}\|(\psi(h) - h)\|$, *where*

(i) M_{hr} *is the spectral norm of* $\frac{\{I-\phi'(h)\}^{-1}}{1-r}$

(ii) $\epsilon_h = \frac{r}{\|(I-\phi'(h))^{-1}\|(M_1+M_2)}$

Proof sketch. Let $h, \hat{h} \in H \subset \mathrm{R}^n$. Since, H is convex, the line segment connecting h and $\hat{h}$, denoted by $\gamma(h, \hat{h})$, is contained within $\mathcal{H}$. Since ψ is differentiable on H, we can apply the mean value theorem component-wise along $\gamma(h, \hat{h})$. Consequently for each $i \in \{1, 2, \ldots, n\}$ there exists $z^{(i)} \in \gamma(h, \hat{h})$ such that

$$\{\psi_i(\hat{h}) - \psi_i(h)\} = (\{\psi'(z^{(i)})(\hat{h} - h)\})_i, \tag{4}$$

where $\psi'(z^{(i)})$ is the Jacobian of the logit mapping ψ evaluated at $z^{(i)}$. Using the fact that the SUE path flow is a fixed point of the logit mapping, i.e., $\psi(\hat{h}) = \hat{h}$, we substitute this into Equation (4) yielding,

$$\{\hat{h}_i - \psi_i(h)\} = \{\psi'(z^{(i)})(\hat{h} - h)\}_i = \{\phi'(z^{(i)})(\hat{h} - h)\}_i. \tag{5}$$

However, we do not have information about $z^{(i)}$ while calculating the bounds at h. Therefore, using the exact form of Taylor's theorem with remainder, we bound the difference $\|\phi'(z^{(i)}) - \phi'(h)\|$. For each component i, we obtain the following bound:

$$|\{\phi'(z^{(i)})(\hat{h} - h)\}_i - \{\phi'(h)(\hat{h} - h)\}_i| \leq (M_1 + M_2)\|\hat{h} - h\|^2. \tag{6}$$

Combining (5) and (6), and using techniques from linear algebra, we arrive at the following inequality:

$$\|(\hat{h} - h)\| - (M_1 + M_2)\|(I - \phi'(h))^{-1}\| \|\hat{h} - h\|^2 \leq \|(I - \phi'(h))^{-1}(\psi(h) - h)\|, \tag{7}$$

where I is the identity matrix and $\phi'(h)$ is the Jacobian of ϕ evaluated at h. Finally, by choosing h sufficiently close to $\hat{h}$, we can show that for every $r > 0$, there exists an ϵ_h such that whenever $\|h - \hat{h}\| < \epsilon_h$, the following holds:

$$\|(\hat{h} - h)\| \leq \frac{\|(I - \phi'(h))^{-1}\|}{1 - r}\|(\psi(h) - h)\|. \tag{8}$$

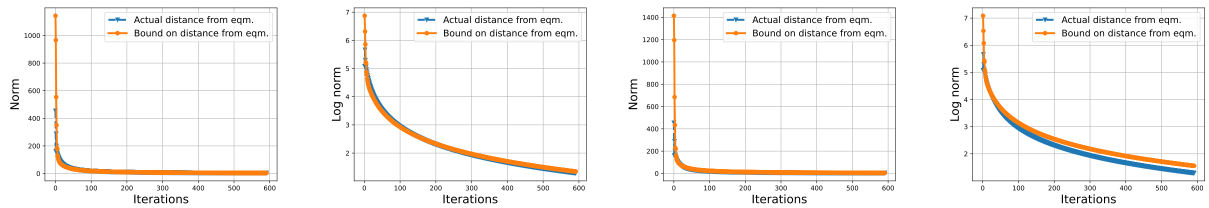
This completes the proof. The ϵ_h is set in such a way so that the quadratic term, $\|\hat{h} - h\|^2$ in Inequality (7) linearises to $r\|\hat{h} - h\|^2$. The exact value of ϵ_h depends on h and is given in the theorem statement.

3 RESULTS AND DISCUSSION

We now provide some insights into the applicability of our result. Theorem 1 establishes that for any path flows h sufficiently close to equilibrium, the distance between the given feasible solution and the equilibrium solution is upper bounded by a function of h. Moreover, the parameter r allows control over how far the bounds begin to hold from equilibrium. To illustrate the tightness of our bound, we use the Sioux Falls test network. The results are shown in Figures 1 for two values of r for a random OD pair. We note from Theorem 1 that for larger values of r, the bound applies from a greater distance from equilibrium but tends to be weaker. Conversely, for smaller r, the bound holds closer to equilibrium and becomes tighter. At equilibrium, the bound is tight with the parameter r set to zero. The plots depict how $\|\hat{h} - h\|$ and $M_{hr}\|\psi(h) - h\|$ evolve over iterations. The dispersion parameter θ was set to 0.5, and the path set was generated using the ten shortest paths for each OD pair. Figure 1 shows the values of $\|\hat{h} - h\|$ and $M_{hr}\|\psi(h) - h\|$ on the actual magnitude as well as on a logarithmic scale. It is important to note that while r can be increased (as per Theorem 1) to make the bound valid from earlier iterations, even with $r \approx 0$, the bound remains close to tight at very early iterations, demonstrating the strength of our result.

We now discuss how our results can be applied in practice. When used as a stopping criterion for a traffic assignment subproblem, these upper bounds can provide valuable insights into how the equilibrium solution might behave for each subproblem. Decision-makers can use these bounds to determine an appropriate stopping criterion, enabling more traffic assignment problems to be solved efficiently as subproblems within larger bi-level network optimization frameworks. Furthermore, an algorithm could be designed to implicitly use these bounds to guide the search towards the SUE solution, utilizing the Jacobian of the logit mapping in the search direction. Finally, our results complement the numerical findings of the gap measure when path flows stabilize with a given percentage of the error in equilibrium flows.

However, we acknowledge a significant limitation of the applicability of these bounds. Without a global bound, it is not possible to determine the iteration from which the local bound becomes applicable. Extending the local bounds to global bounds is not straightforward, as the parameter r in Theorem 1 cannot be arbitrarily large. While empirical studies may help to estimate when these bounds are reliable, developing a global bound would provide greater confidence in their application. Furthermore, the computational effort in calculating these bounds must be carefully considered. If the computation is too costly, it defeats the purpose of applying the bounds rather than performing additional iterations to improve path flows. For instance, in the example shown in Figure 1, the bounds computation took around 0.07s, while each iteration of method of successive averages to improve path flows took around 0.24s. Future work should make efforts to improve the computational effort spent in computing these bounds.



$r = 0.01$ *(actual scale)* $r = 0.01$ *(log scale)* $r = 0.2$ *(actual scale)* $r = 0.2$ *(log scale)*

Figure 1 – *Bound in the Sioux Falls network OD pair* $(8, 11)$ *for two different values of* r

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