A Multi-objective User Equilibrium of Time Loss in Congestion and Time Surplus

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1 INTRODUCTION

Modelling route choice behaviour in a road network has been studied for decades since the discovery of Wardrop (1952)'s principles. The attractiveness of the well known User Equilibrium (UE) based on Wardrop's first principle lies in its simplicity and its consistency with Nash equilibrium in game theory. The UE model has its limitations due to the four key assumptions behind Wardrop's first principle. The Stochastic User Equilibrium (SUE) model, proposed by Daganzo & Sheffi (1977), has become the mainstream alternative modelling method to the standard deterministic UE (DUE) model, addressing some of the limitations. Until today, the value of time (VOT) has been modelled as a coefficient of one of the parameters in the generalised cost (GC) function in DUE or utility function in SUE. The norm in modelling and in practice is to assume that users can be classified into multi-user classes and each user class will have their own VOT. VOTs have always been viewed as constants within a GC or utility function, which is presented as a linear function of time. However, empirical evidence shows that drivers' VOTs might vary when they are faced with different driving conditions (Wardman & Ibanez, 2012). In practice, extensive efforts have been devoted to estimating VOTs and congestion-dependent VOT (CVTT) multipliers (Batley & Mackie, 2020, Batley *et al.*, 2022) but the estimates are much higher than Wardman & Ibanez (2012)'s. A research gap has been identified in how to model the effect of congestion on route choice behaviour. Is it possible to apply CVTT multipliers in a conventional GC function? Or do we need a new modelling framework?

We postulate that VOT is non-linear in nature, due to asymmetric preferences of users when they are faced with route choices. De Borger & Fosgerau (2008) have shown that users' willingness to pay (WTP) to shorten their journey time might not be the same as their willingness to accept (WTA) longer journey time. The asymmetry of WTP and WTA has led to a need to model the non-linear nature of VOT. Wang & Ehrgott (2013) are the first to model the non-linear nature of VOT implicitly by introducing indifference curves to represent the maximum time drivers are willing to spend for a given monetary cost for each origin-destination pair. All users will maximise their time surplus (TS) defined as the maximum time they are willing to spend minus the actual time spent. A Time Surplus maximisation Bi-objected User Equilibrium (TSmaxBUE) is achieved when all used paths have equal and higher TS than any unused paths. By nature an indifference curve, which can be concave or convex, would be a strictly decreasing function as a user must want to spend less time if he/she has to pay more. Ding *et al.* (2023) propose a more general Status-quo Dependent User Equilibrium (SDUE) model, which can be reduced to a TSmaxBUE model. The SDUE model is intuitively based on the likeliness of path-switching behaviour. In essence, each user is checking the ratio of potential increase/saving in monetary cost to potential travel time saving/increase against his/her own critical VOT as a mechanism to decide if it is *worthwhile* to switch to an alternative path. If all users find their current paths acceptable, an equilibrium is reached. In this way, the meaning of VOT has taken on a new virtual representation as the slope of any straight line connecting any two alternatives along an indifference curve. The upper and lower bounds of VOT can be determined for any given indifference curve and they become the boundary VOTs.

In the present study, we focus on the non-linear nature of VOT and propose a new multi-objective user equilibrium (MUE) model, offering a new perspective of the trade-offs between time loss in congestion and time surplus. It enables not only the modelling of the variation of VOT among users but also the variation of VOT for the same user for different route choices as well as the components of their journey time on each route. We also look into the implications on the existence of equivalent CVTT multipliers for time loss in congestion under MUE conditions.

2 The Model

Empirical Evidence Yamamoto *et al.* (2002) conducted an innovative study in Japan applying data mining algorithms to determine decision trees representing route choice behaviour. Their findings show that expected minimum, average and maximum travel time, are considered by travellers in their decision trees, although not all travellers might have included all three factors. The consideration of average and maximum travel time is consistent with the inherent multi-objective equilibrium model structure of TSmaxBUE. TS is effectively maximum travel time minus average travel time.

In contrast, the influence of the expected minimum travel time, which has been found to be significant on route choice behaviour in Yamamoto *et al.*'s study, has never been formalised in traffic assignment models. Based on the WTP concept, a user might be willing to pay for having a chance to have a lower minimum expected travel time as well as a lower maximum expected travel time. Such desire might vary among travellers. That is, some travellers might be willing to pay for a lower minimum while others might be more willing to pay for a lower maximum.

Modelling Time Loss in Congestion versus Time Surplus We assume that each individual would have his/her own minimum expected travel time based on a desired level of service (LOS). This minimum expected travel time might vary among individuals and even for the same individual it might vary by time of the day. For instance, during the peak hours, a traveller might think that 25 minutes (LOS C) is the most ideal (as he/she might have accepted that a certain level of congestion during the peak is unavoidable). For the same person, he/she might have a minimum expected 15-minute travel time (LOS A) during the off-peak. In Yamamoto *et al.* (2002)'s experiment, this minimum expected travel time was found to be 8 minutes for the drivers choosing between the second Shin-Mei Expressway versus the new north route.

We define Time Loss (TL) in congestion as the actual time spent minus the minimum expected travel time. For each alternative route, the higher the monetary cost, the lower the minimum expected travel time would be. When we couple the new TL concept with the TS concept, a new MUE model is emerged. While travellers would like to maximise their TS, they would be minimising their TL at the same time. Their route choices will depend on how they trade off between combinations of TL and TS on each route.

Mathematical formulation Let D_p denote travel demand, i.e. the number of vehicles travelling from origin O to destination D, (OD) pair $p \in P$, where P is a set of all OD pairs. Let $c_a(f_a)$ denote a cost function of link a that depends on link flow f_a. Let $\mathbf{f} = (f_1, f_2, \ldots, f_{|A|})$ denote a vector of link flows.

Let $F = (F_1, F_2, \ldots, F_{|K|})$ denote a vector of path flows, where K is a set of simple paths bewteen all OD pairs of a graph $G(N, A)$. $T_k(\mathbf{f}) = \sum_{a \in A} \delta_a^k c_a(f_a)$, where δ_a^k equals 1 if link a is along path k, or 0 otherwise. And, τ_k denotes the toll on path k, $\tau_k = \sum_{a \in A} \delta_a^k \omega_a$, where ω_a is a constant toll of link a.

To consider the trade-offs between TL and TS, we now define the following notations. Let $T_k^{\max}, T_k^{\min} : \mathbb{R} \to \mathbb{R}$ be strictly decreasing functions, meaning that $T_k^{\max}(\tau_k^1) < T_k^{\max}(\tau_k^2)$ if $\tau_k^1 > \tau_k^2$, and similarly for $T_k^{\min}$. Let $T_k^{\max}$ be a non-linear function that depends on path toll representing the maximum time a road user is willing to spend. Let $T_k^{\min}$ be the minimum expected travel time on path k, which is the sum of the minimum expected travel time on each link along the path, i.e. $T_k^{\min} = \sum_{a \in A} \delta_a^k t_a^{\min}$, where $t_a^{\min}$ is the minimum expected travel time on link a.

Each user will maximise his/her time surplus, TS_k, which equals the maximum time he/she is willing to spend minus the actual time spent, as shown in Eqn (1),

$$\max TS_k(\mathbf{f}) = T_k^{\max}(\tau_k) - T_k(\mathbf{f}), \quad \forall k \in K_p, \forall p \in P. \tag{1}$$

Each user will also minimise his/her time loss, TL_k, which equals the actual time spent minus the minimum expected travel time, as shown in Eqn (2),

$$\min TL_k(\mathbf{f}) = T_k(\mathbf{f}) - T_k^{\min}(\tau_k), \quad \forall k \in K_p. \tag{2}$$

We further assume that TS and TL are compensatory, i.e. for each unit time gain of TS, there exists an equivalent increase of TL a traveller is willing to accept. Each traveller is characterised by his/her own indifference curves for $T_k^{\max}$ and $T_k^{\min}$, as well as his/her weightings for TS and TL, represented by $\theta \in [0, 1]$, forming the objective function of Eqn (3),

$$\min Z_k(\mathbf{f}) = \theta TL_k(\mathbf{f}) - (1 - \theta) TS_k(\mathbf{f}). \tag{3}$$

Substituting Eqn (1) & (2) into (3) gives Eqn (4),

$$\min Z_k(\mathbf{f}) = T_k(\mathbf{f}) - \left(\theta T_k^{\min} + (1 - \theta) T_k^{\max}\right). \tag{4}$$

At equilibrium, all used paths will have equalised Z, e.g. $Z_k(\mathbf{f}) = Z_j(\mathbf{f})$.

Definition 1. Based on Ding *et al.* (2023)'s SDUE formulation: The travellers between OD pair $p \in P$ would stick to their current path $k \in K$ if and only if Eqn (5) holds, i.e. path k is an acceptable path. Otherwise, the travellers would switch to other alternatives between OD pair p to reduce their travel costs.

$$\tau_k^p - \tau_j^p \leq \alpha(T_j^p(\mathbf{f}) - T_k^p(\mathbf{f})) + \epsilon_k^p, \ \forall j \in K_p, \ p \in P, \tag{5}$$

where α and ϵ_k^p are route choice parameters, respectively, $\alpha \in (\alpha^{LB}, \alpha^{UB})$ with $0 \leq \alpha^{LB} \leq \alpha^{UB}$, and $0 \leq \epsilon_k^p \leq \epsilon_{k*}^p = \left(\alpha^{UB} - \alpha^{LB}\right) T_k^p$.

Ding *et al.* (2023) have shown that the SDUE formulation can be reduced to Wang & Ehrgott (2013)'s TSmaxBUE model when $\epsilon_k^p = 0$. By rearranging Eqn (5), we have Eqn (6),

$$\frac{\tau_k^p - \tau_j^p}{T_j^p(\mathbf{f}) - T_k^p(\mathbf{f})} \leq \alpha, \tag{6}$$

where $\alpha \in (\alpha^{LB}, \alpha^{UB})$ with $0 \leq \alpha^{LB} \leq \alpha^{UB}$.

If we look at the actual travel time and split it into two components, by dropping p for simplicity from now on, if a VOT multiplier β_k for path k exists, the conventional model assumes that travellers would minimise a VOT-multiplier-weighted time function T^w in Eqn (7),

$$T_k^w(\mathbf{f}) = T_k^{\min} + \beta_k \left(T_k(\mathbf{f}) - T_k^{\min}\right). \tag{7}$$

Since all used paths have equalised $Z(\mathbf{f})$, i.e. $Z_k(\mathbf{f}) = Z_j(\mathbf{f})$. If equivalent multipliers exist, we will also have equalised $T_k^w(\mathbf{f})$, i.e. $T_k^w(\mathbf{f}) = T_j^w(\mathbf{f})$. Thus, we can find the VOT multiplier β_k for path k by solving,

$$T_k^w(\mathbf{f}) = Z_k(\mathbf{f}) \tag{8}$$

$$T_k^{\min} + \beta_k \left(T_k(\mathbf{f}) - T_k^{\min}\right) = T_k(\mathbf{f}) - \left(\theta T_k^{\min} + (1-\theta) T_k^{\max}\right) \tag{9}$$

$$\beta_k = \frac{T_k(\mathbf{f}) - (1-\theta)\left(T_k^{\min} + T_k^{\max}\right)}{T_k(\mathbf{f}) - T_k^{\min}}. \tag{10}$$

This means that we can always find the equivalent value of β_k satisfying $T_k^w(\mathbf{f}) = Z_k(\mathbf{f})$ as well as β_j satisfying $T_j^w(\mathbf{f}) = Z_j(\mathbf{f})$, but β_k, β_j are path-dependent even for the same user.

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